5^{th} Lecture

Functions of Two and More Variables, Domains of Definition, Graphs, Limits

n-dimensional Euclidean Space ($n \in \mathrm{N}$).

A space, consisting of all n-tuples (ordered) of real numbers, where the distance between two points $X = [x_1, x_2, \cdots, x_n]$, $Y = [y_1, y_2, \cdots, y_n]$ is defined as

$$d(X,Y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

is called **n-dimensional Euclidean space** and denoted by E_n.
(If $n = 1$, then $d(X,Y) = |x_1 - y_1|$.) It can be proved, that for every natural n and for any triplet of points $X, Y, Z \in E_n$, it holds:

1. $d(X,Y) \geq 0, \quad d(X,Y) = 0 \Longleftrightarrow X = Y$,
2. $d(X,Y) = d(Y,X)$,
3. $d(X,Y) \leq d(X,Z) + d(Z,Y)$.

The function $d : E_n \times E_n \to \mathbb{R}$ is called the **metric** on E_n and the couple (E_n, d) is then the **metric space**.

Let $X_0 \in E_n$ and $\varepsilon > 0$. The set

$$N_\varepsilon(X_0) = \{X \in E_n : d(X, X_0) < \varepsilon\}$$

is called the **ε-neighbourhood of** X_0.

It is clear, that $N_\varepsilon(X_0)$ is an open interval, if $n = 1$, a disk, if $n = 2$ and a sphere, if $n = 3$.

Let $M \subset E_n$. A point $X_0 \in M$ is called an **interior point** of M, if there exists $\varepsilon > 0$ such that $N_\varepsilon(X_0) \subset M$.

The set of all interior points of a set M is called the **interior** of the set M.

A set M is said to be **open**, if it consists of its interior points.

A point $X_0 \in E_n$ is called a **boundary point** of a set $M \subset E_n$, if each $N_\varepsilon(X_0)$ contains at least one point which belongs to the set M and at least one point which does not belong to the set M.

The set of all boundary points of a set M is called the **boundary** of the set M.

A set M is said to be **closed**, if it contains its boundary.

Remark. It is accustomed that coordinates of points in E_2 and E_3, instead of $[x_1, x_2]$ and $[x_1, x_2, x_3]$, are denoted by $[x, y]$ and $[x, y, z]$.

Example 1. Let

$$M = \{[x, y] : 0 \le x \le 1,\ 0 < y < 1\} \subset E_2.$$

Find the interior and the boundary of the set M.

Example 2. Let

$$M = \{[x, y, z] : 0 < x < 1,\ 0 < y < 1,\ z = 0\} \subset E_3.$$

Find the interior and the boundary of the set M.

Let $M \subset E_n$. Then

- The set M is called **connected**, if each pair of its points can be joined by a simple curve, lying in the set M.
- The set M is called **bounded**, if there exists a real number $R \in \mathbb{R}$ and a point $X_0 \in E_n$ such that for each $X \in M :\ d(X, X_0) < R$.

Example 3. Let

a) $M = \{[x, y] : \frac{x^2}{4} + y^2 < 1\} \subset E_2,$

b) $M = \{[x, y] : x.y < 1,\ x \ge\ \ y \ge\} \subset E_2,$

c) $M = \{[x, y, z] : x^2 + y^2 < 4,\ 0 < z < 1\} \subset E_3,$

d) $M = \{[x, y, z] : x.y.z > 0\} \subset E_3.$

Find the interior and the boundary of the set M and find out whether the set M is open, closed, bounded and connected.

Functions of Several Variables.

Let $M \subset E_n,\ M \ne \emptyset$. A mapping f which assigns to each $X = [x_1, x_2, \cdots, x_n] \in M$ exactly one real number is called a **real function of n real variables**. M is its **domain of definition**, $M = D(f)$.

It is written:

$$y = f(X) = f(x_1, x_2, \cdots, x_n),$$

if $n = 2 :\ z = f(x, y)$, if $n = 3 :\ u = f(x, y, z)$.

Example 4. Sketch domains of definition for the following functions:

a) $f(x, y) = \frac{1}{x - 2y}, \quad D(f) = \{[x, y] : x - 2y \ne 0\} \subset E_2,$

b) $f(x, y) = \ln xy, \quad D(f) = \{[x, y] : xy > 0\} \subset E_2,$

c) $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}, \quad D(f) = \{[x, y, z] : x^2 + y^2 + z^2 \le 1\} \subset E_3,$

d) $f(x,y,z) = z.\arcsin(x+y), \quad D(f) = \{[x,y,z] : -1 \le x+y \le 1\} \subset E_3.$

Some of significant concepts, like, for example, boundedness (boundedness from below and boundedness from above), maximum, minimum and operations on functions are defined analogously as in the real case ($n = 1$).

Example 5. Let

$$M = \{[x,y,z] : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\} \subset E_3$$

and let $f(x,y,z) = x + y + 3z$. Show, that the function f is bounded on M and find the minimum and the maximum of f on M.

Graph of a Function of Several Variables.

Let f be a function of n variables ($D(f) \subset E_n$). Graph of the function f is a set $G(f) \subset E_{n+1}$ of all ordered $(n+1)$-tuples $[x_1, x_2, \cdots, x_{n+1}]$ such that $[x_1, x_2, \cdots, x_n] \in D(f)$ and $x_{n+1} = f(x_1, x_2, \cdots x_n)$. Therefore

$$G(f) = \{[x_1, x_2, \cdots, x_n, x_{n+1}] :\ X = [x_1, x_2, \cdots, x_n] \in D(f),\ x_{n+1} = f(X)\} \subset E_{n+1}.$$

If $n = 2$ ($D(f) \subset E_2$), then

$$G(f) = \{[x,y,z] :\ [x,y] \in D(f),\ z = f(x,y)\} \subset E_3.$$

Example 6. Find domains of definition and sketch graphs of the following functions:

a) $f(x,y) = 1 - 2x - 2y,$

b) $f(x,y) = \dfrac{1}{2x^2 + 3y^2},$

c) $f(x,y) = \sqrt{1 - x^2 - y^2}.$

If $G(f) \subset E_3$, then its orthogonal projection onto the plane P_{xy} coincides with the domain of definition $D(f)$. Each straight line parallel to O_z intersects $G(f)$ at most at one point. In fact, this is the necessary condition for a surface in E_3, to be the graph of a function of two variables. Just this is the reason, why, for example, any whole spherical surface cannot be graph of a function of two variables.

Example 7. Find domains of definition, sketch graphs and determine quadric surfaces whose parts the following graphs are.

a) $f(x,y) = \sqrt{1-y^2},$	*b)* $f(x,y) = \sqrt{x^2-9},$
c) $f(x,y) = -\sqrt{3-x},$	*d)* $f(x,y) = \sqrt{4+y},$
e) $f(x,y) = -\sqrt{y-x^2},$	*f)* $f(x,y) = \sqrt{x^2+y^2-1},$
g) $f(x,y) = \sqrt{y^2-x^2},$	*d)* $f(x,y) = -\sqrt{4+x^2+y^2}.$

Limit of a Function of Several Variables.

Definition of the limit is the same, as for functions of one real variable.

We suppose, that a function f of n variables is defined in a neighbourhood of a point $A = [a_1, a_2, \cdots, a_n]$, excepting possibly the point A.

It is said, that a real number b is the **limit** of the function f at the point A, if for each $\varepsilon > 0$ there exists $\delta > 0$, such that $X \in O_\delta(A),\ X \neq A$, implies $|f(X) - b| < \varepsilon$. Limit of f at A is denoted by $\lim\limits_{X \to A} f(X)$, therefore

$$\lim_{X \to A} f(X) = b \overset{def}{\iff} \forall\, \varepsilon > 0\ \exists\, \delta > 0\ :\ 0 < d(X, A) < \delta \Rightarrow |f(X) - b| < \varepsilon$$

In the case if f is a function of two variables and $A = [x_0, y_0]$ it can be written in the form:

$$\lim_{[x,y] \to [x_0,y_0]} f(x,y) = \lim_{\substack{x \to x_0 \\ y \to y_0}} f(x,y) = b \overset{def}{\iff}$$

$$\overset{def}{\iff} \quad \forall\, \varepsilon > 0\ \exists\, \delta > 0\ :\ 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x,y) - b| < \varepsilon$$

Analogously in the case of a function of three variables.

Example 8. By means of the definition prove, that $\lim\limits_{[x,y] \to [1,2]} (x + y) = 3$.

Example 9. By means of the definition prove, that $\lim\limits_{[x,y] \to [0,0]} \dfrac{xy}{x^2 + y^2}$ does not exist.

Let us suppose, that for two functions of n variables f_1 a f_2 it is valid:

$$\lim_{X \to A} f_1(X) = b_1 \quad \text{and} \quad \lim_{X \to A} f_2(X) = b_2.$$

Then

1. $\lim\limits_{X \to A} (c_1 f_1(X) + c_2 f_2(X)) = c_1 b_1 + c_2 b_2$, for any real constants c_1, c_2.
2. $\lim\limits_{X \to A} (f_1(X).f_2(X)) = b_1.b_2$,
3. $\lim\limits_{X \to A} \dfrac{f_1(X)}{f_2(X)} = \dfrac{b_1}{b_2}$, for $b_2 \neq 0$.

Improper limits for functions of n variables (at a proper point) are also defined in the same way as in the one-dimensional case. Let us introduce definition of the improper limit $+\infty$ at a proper point $A \in E_n$ (Definition for $-\infty$ is analogical):

$$\lim_{X \to A} f(X) = \infty \overset{def}{\iff} \forall\, K > 0\ \exists\, \delta > 0\ \ 0 < d(X, A) < \delta \implies f(X) > K$$

Example 10. By means of the definition prove, that $\lim\limits_{[x,y] \to [0,0]} \dfrac{1}{x^2 + y^2} = \infty$.