# $12^{th}$ Lecture

## Triple Integrals in Cylindrical and Spherical Coordinates.

Similarly as in the plane, the rectangular coordinate system in three dimensional space is not appropriate for all types of problems. There are circumstances in which other systems are more convenient. In some problems, concerning triple integrals over some special types of domains of integration, we will use two of them, the **Cylindrical and Spherical coordinate systems**.

**The Cylindrical Coordinate System in $E_3$.**

In cylindrical coordinates the position of a point $M \in E_3$ is specified by an ordered triplet of real numbers, $[r, \varphi, u]$, where $[r, \varphi]$ are polar coordinates of the projection of the point $M$ onto the plane $R_{xy}$ and $u$ is its $z$-coordinate ($u = z$). It follows, that $r \geq 0$, $\varphi \in \langle 0, 2\pi)$ and $u \in \mathbb{R}$.

When we use both cylindrical and Cartesian coordinates in space ($M = [r, \varphi, u] = [x, y, z]$), then the two sets of coordinates are related by the equations:

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = u$$

Example 1. Replace the following cylindrical equations by equivalent Cartesian equations and identify them:

$$a)\ r = 1, \qquad b)\ \varphi = \frac{\pi}{2}, \qquad c)\ u = -1, \qquad d)\ u = r.$$

Example 2. Graph the sets of points in $E_3$, whose cylindrical coordinates satisfy the following conditions:

1. $0 \leq r \leq 1, \quad 0 \leq u \leq 1,$
2. $r = 2, \quad 0 \leq \varphi \leq \frac{\pi}{4},$
3. $1 \leq r \leq 2, \quad 0 \leq \varphi \leq \frac{\pi}{2}, \quad 1 \leq u \leq 3,$
4. $u \leq 0, \quad (\varphi = 0) \vee (\varphi = \pi).$

The cylindrical coordinates are appropriate mainly to describe solids, enclosed by cylindrical surfaces.

## The Spherical Coordinate System in $E_3$.

In spherical coordinates the position of a point $M \in E_3$ is specified by an ordered triplet of real numbers, $[r, \varphi, \vartheta]$, where $r$ is the distance from the point $M$ to origin, $r = |\overline{MO}|$, $\varphi$ is the polar coordinate of its projection $M'$ onto the plane $R_{xy}$ and $\vartheta$ is the space angle made by line segments $\overline{MO}$ and $\overline{M'O}$. It follows, that $r \geq 0$, $\varphi \in \langle 0, 2\pi)$ and $\vartheta \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$.

It is a convention that $\vartheta$ assumes positive values for positive $z$ and negative values for negative $z$.

When we use both spherical and Cartesian coordinates in space ($M = [r, \varphi, \vartheta] = [x, y, z]$), then the two sets of coordinates are related by the equations:

$$x = r\cos\varphi\cos\vartheta, \qquad y = r\sin\varphi\cos\vartheta, \qquad z = r\sin\vartheta$$

The spherical coordinates are appropriate mainly for describing solids, enclosed by spherical surfaces.

Example 3. Replace the following spherical equations by equivalent Cartesian equations and identify them:

$$a)\ r = 2, \qquad b)\ \varphi = \pi, \qquad c)\ \vartheta = 0, \qquad d)\ \vartheta = \frac{\pi}{4}.$$

Example 4. Graph the sets of points in $E_3$, whose spherical coordinates satisfy the following conditions:

1. $0 \leq r \leq 2, \quad \vartheta = -\frac{\pi}{4}$,
2. $1 \leq r \leq 2, \quad \varphi = \pi$,
3. $r = 1, \quad (\varphi = \frac{\pi}{2}) \vee (\varphi = \frac{3\pi}{2})$,
4. $\frac{\pi}{4} \leq \vartheta \leq \frac{\pi}{2}$.

## Change of Variables in Triple Integrals.

Change of variables in triple integrals is used, similarly as in double integrals, not only in the case of "too complicated" functions, but also (and more frequently) for "too complicated" domains of integration.

Domain of integration is often a space region not bounded by planes, but by some surfaces, for instance by parts of cylindrical or spherical surfaces. Especially in these cases it is simpler to describe the regions by means of cylindrical or spherical, instead of Cartesian, coordinates.

We shall not present the general formula for the transformation of triple integrals, but only two its special forms, for transformation to cylindrical and to spherical coordinates.

**Transformation to Cylindrical coordinates.**

If $\Phi : E_3 \to E_3$ is the transformation from Cartesian coordinates $[x, y, z]$ to cylindrical coordinates $[r, \varphi, u]$, given by formulas

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = u,$$

it can be simply proved, that the Jacobian of this transformation (defined analogously as in the two dimensional case, for transformations from $E_2$ to $E_2$) $D_\Phi(r, \varphi, u) = r$.

Then, if $f(x, y, z)$ is a function, continuous on a region $R \subset E_3$

$$\iiint\limits_R f(x, y, z)dxdydz = \iiint\limits_{R^*} f(r\cos\varphi, r\sin\varphi, u).r\ drd\varphi du,$$

where $R^* = \Phi^{-1}(R)$.

**Transformation to Spherical coordinates.**

If $\Phi : E_3 \to E_3$ is the transformation from Cartesian coordinates $[x, y, z]$ to spherical coordinates $[r, \varphi, \vartheta]$, given by formulas

$$x = r\cos\varphi\cos\vartheta, \qquad y = r\sin\varphi\cos\vartheta, \qquad z = r\sin\vartheta,$$

it can be simply proved, that the Jacobian of this transformation $D_\Phi(r, \varphi, \vartheta) = r^2\cos\vartheta$.

Then, if $f(x, y, z)$ is a function, continuous on a region $R \subset E_3$

$$\iiint\limits_R f(x, y, z)dxdydz = \iiint\limits_{R^*} f(r\cos\varphi\cos\vartheta, r\sin\varphi\cos\vartheta, r\sin\vartheta).r^2\cos\vartheta\ drd\varphi d\vartheta,$$

where $R^* = \Phi^{-1}(R)$.

**Remark 1.** These two transformation formulas for change of variables in triple integrals are applicable only under some conditions on the domain of integration, the region $R$, and properties of the transformation in this region.

Example 5. By means of transformation to cylindrical or spherical coordinates, calculate the following integrals.

1. $\iiint\limits_R zdxdydz$, if $R$ is the set bounded by surfaces $x^2 + y^2 = 4,\ z = 1,\ z = 3$,
2. $\iiint\limits_R \sqrt{x^2 + y^2 + z^2}dxdydz$, if $R$ is the set, bounded by surfaces $x^2 + y^2 + z^2 = 1$, $x = 0,\ y = 0,\ z = 0$, in the first octant,
3. $\iiint\limits_R (x^2 + y^2)dxdydz, \quad R = \{[x, y, z] : z \geq 0,\ 4 \leq x^2 + y^2 + z^2 \leq 9\}$,
4. $\iiint\limits_R zdxdydz, \quad R = \{[x, y, z] : x^2 + y^2 + z^2 \leq z\}$,
5. $\iiint\limits_R zdxdydz$, if $R$ is the set bounded by surfaces $x^2 + y^2 = 2y,\ x^2 + y^2 + z^2 = 4$, $z \leq 0$,

6. $\iiint\limits_R dxdydz, \quad R = \{[x,y,z] : x^2+y^2 \leq 1,\ x \geq 0,\ 0 \leq z \leq 6\}$,

7. $\iiint\limits_R (x^2+y^2)dxdydz, \quad R = \{[x,y,z] : x^2+y^2 \leq 2z,\ y \leq 0,\ z \leq 2\}$,

8. $\iiint\limits_R z\sqrt{x^2+y^2}dxdydz$, if $R$ is the set bounded by surfaces $x^2+y^2 = 2x,\ y = 0,$ $z = 0,\ z = 1,\ y \geq 0.$

**Applications of Triple Integrals in Cylindrical and Spherical Coordinates.**

Example 6. Find the volume of the solid bounded by surfaces

1. $x = x^2+y^2, \quad z = 1,$
2. $x^2+y^2+z^2 = 4, \quad x^2+y^2 = z^2,$
3. $x^2+y^2 = 2x, \quad x^2+y^2+z^2 = 4,$
4. $x^2+y^2+z^2 = 1, \quad x^2+y^2+z^2 = 4, \quad x^2+y^2 = z^2, \quad z \geq 0.$

Example 7. Find the total mass of the solid bounded by surfaces

1. $x^2+y^2 = 9, \quad z = 1, \quad z = 4$, if its density is $\sigma(x,y,z) = x^2+y^2$,
2. $z = \sqrt{x^2+y^2}, \quad 2z = x^2+y^2$, if its density is $\sigma(x,y,z) = z^2$.

Example 8. Find coordinates of the centroid of the solid bounded by surfaces

1. $x^2+y^2 = 4, \quad z = x^2+y^2, \quad z = 0,$
2. $z = x^2+y^2, \quad z = \frac{1}{2}(x^2+y^2+1).$

Example 9. Find coordinates of the centroid of the solid enclosed by the sphere $x^2+y^2+z^2 = 1$, that lies between the half planes $\varphi = -\frac{\pi}{3},\ z \geq 0, \quad \varphi = \frac{\pi}{3},\ z \geq 0.$