Linear Second Order Homogeneous Differential Equations with Constant Coefficients

Differential equation of the type

$$y''+p_1y'+p_2y=g(x) \quad (1),$$

where p_1 and p_2 are real numbers and $g(x)$ is a function continuous on an interval (a,b) and moreover $g(x)\not\equiv 0$ on (a,b) is called a **non-homogeneous linear differential equation of the 2nd order with constant coefficients.**

A special case we obtain for $g(x)\equiv 0$:

$$y''+p_1y'+p_2y=0 \quad (2),$$

which is called a **homogeneous linear differential equation of the 2nd order with constant coefficients.**

If a_0, a_1 are arbitrary real numbers, it can be proved that there exists just one solution of the equation (1) satisfying initial conditions: $y(x_0)=a_0$, $y'(x_0)=a_1$ $(x_0\in R)$. The same is valid for the equation (2).

Linear dependence and independence of solutions.

Two solutions of the equation (2) are **linearly dependent** on R, if there exists such a number $k\in R$, that $\forall x\in R: y_1=ky_2$. If solutions y_1, y_2 of the equation (2) are not linearly dependent, they are called **linearly independent**.

Example 1. a) Show, that functions $y_1=e^{3x}$, $y_2=5e^{3x}$ are linearly dependent solutions of the equation $y''-5y'+6y=0$ on R

b) Show, that functions $y_1=e^{x}$, $y_2=1$ are linearly independent solutions of the equation $y''-y'=0$ on R.

Let functions y_1 and y_2 are differentiable on an interval J. Then determinant created for each $x\in J$ in the following way: $\begin{vmatrix} y_1(x), & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}$ is called their **Wronskian** on J and denoted $W(y_1,y_2)$, or $W(x)$. Therefore: $W(x)=\begin{vmatrix} y_1(x), & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}$ is Wronskian, or Wronsky's determinant of functions y_1 and y_2 on J.

Let functions y_1 and y_2 are solutions of the equation (2) on J. Functions y_1 and y_2 are linearly independent on J if $W(y_1,y_2)\neq 0$, for each $x\in J$.

Any couple of two linearly independent solutions of the equation (2) is called the **fundamental system of solutions** of (2).

General solution: If y_1, y_2 create a fundamental system of solutions of the equation (2), then the general solution of (2) is:

$$y = c_1 y_1 + c_2 y_2 ,$$

where c_1 and c_2 are arbitrary constants.

<u>Example 2.</u> By means of Wronskian verify, that functions $y_1 = e^x$ and $y_2 = xe^x$ create a fundamental system of solutions of the equation $y'' - 2y' + y = 0$ on R. Write the general solution.

Suppose, that one particular solution of the equation (2) is a function $y = e^{rx}$, $r \in R$. Then: $y' = re^{rx}$, $y'' = r^2 e^{rx}$, it follows, that $y = e^{rx}$ is a solution of (2), if $e^{rx}\left(r^2 + p_1 r + p_2\right) = 0$, therefore $r^2 + p_1 r + p_2 = 0$. This equation (algebraical) is called a **characteristic equation** of the equation (2).

It follows, that a function $y = e^{rx}$ is a solution of the equation (2), if the number r is a root of its characteristic equation.

Because the characteristic equation is a quadratic equation, there are three possibilities:

a) there exist two distinct real roots
b) there exists a double real root
c) there exists a couple of complex conjugate roots

a) Let the characteristic equation has two real roots r_1, r_2 , $r_1 \neq r_2$. Then it can be proved, that these functions: $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are both solutions of (2), $W(y_1, y_2) = e^{(r_1 + r_2)x}(r_2 - r_1) \neq 0$, $\forall x \in R$, hence y_1 and y_2 are linearly independent and thus they create a fundamental system of solutions, it means: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is a general solution of (2), $c_1, c_2 \in R$

<u>Example 3.</u> Solve the equation: $y'' - 4y' + 3y = 0$

b) Let the characteristic equation has a double real root $r_1 = r_2 = r$. Then it can be proved, that functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are solutions of (2), $W(y_1, y_2) = e^{2rx} \neq 0$, $\forall x \in R$. It follows, that functions $y_1 = e^{rx}$, $y_2 = xe^{rx}$ create a fundamental system, and $y = c_1 e^{rx} + c_2 x e^{rx}$, $c_1, c_2 \in R$ is a general solution.

<u>Example 4</u>. Solve the equation $y'' + 6y' + 9y = 0$

c) Let the characteristic equation has a couple of complex conjugate roots $r_1 = a + ib$, $r_2 = a - ib$, $a, b \in R$, $b \neq 0$. Then the complex function $y = e^{r_1 x} = e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ satisfies the equation (2) on R. Then it can be proved that the real and the imaginary parts of the function y, it means $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ are real solutions of (2) and because $W(y_1, y_2) = be^{2ax} \neq 0$, $\forall x \in R$, y_1 and y_2 are linearly independent and create a fundamental system. Therefore $y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$, $c_1, c_2 \in R$ is a general solution.

Example 5. Solve the equation $y'' - 6y' + 13y = 0$

Example 6. Find particular solution of the differential equation, satisfying given initial conditions:

a) $y'' + 3y' = 0$, $y(0) = 2$, $y'(0) = 3$
b) $y'' + 4y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$
c) $y'' - 4y' + 4y = 0$, $y(0) = -1$, $y'(0) = 0$

Vibrations problem plays an important role in modern engineering and physics. There are many cases when vibrations are described by linear differential equations of the 2^{nd} orders, having constant coefficients:

Example 7. Suppose that a moving body of the mass m is under the action of a force directed toward the state of equilibrium, the magnitude of the force being proportional to the deviation of the state. If we neglect the resistance of the medium, this motion is said to be a simple harmonic motion. Find its law.

Solution:
If the distance from the body to the state of equilibrium is denoted by s, then the force $F = -as$, a being a positive constant. According to the Newton's 2^{nd} law of motion:

$$m\frac{d^2s}{dt^2} = -as \Rightarrow m\frac{d^2s}{dt^2} + as = 0 \quad (ms'' + as = 0)$$

If we denote $k^2 = \frac{a}{m}$, we obtain $s'' + k^2 s = 0$

It follows that $s = c_1\cos kt + c_2\sin kt$, $c_1, c_2 \in R$. It means that s is a periodic function of time t. Its period is $T = \frac{2\pi}{k}$.