Integration of Some Rational, Irrational and Trigonometric Functions

One of the most important classes of elementary functions, whose antiderivatives can be found in comparatively simple way and always are elementary functions, are rational functions.

Consider integrals of the type

$$\int \frac{P(x)}{x^2 + px + q} dx$$

where $P(x)$ is a polynomial, $p, q \in R$. If the degree of the polynomial $P(x)$ is greater than 1, the division of $P(x)$ by $x^2 + px + q$ results in a polynomial $Q(x)$ and a polynomial $ax + b$, as the remainder. Consequently

$$\frac{P(x)}{x^2 + px + q} = Q(x) + \frac{ax + b}{x^2 + px + q}$$

The integration of the polynomial $Q(x)$ does not present any difficulties and hence the problem reduces to integrating a fraction $\frac{ax+b}{x^2 + px + q}$, if $a^2 + b^2 \neq 0$:

Each integral of that type can be transformed to the one of following basic types:

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \quad (a > 0)$ (tablet integral)
2. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right) = \frac{1}{2a}\left(\ln|x-a| - \ln|x+a| \right) + C = \frac{1}{2a} \ln\left| \frac{x-a}{x+a} \right| + C$
3. $\int \frac{x}{x^2 \pm a^2} dx = \frac{1}{2} \int \frac{2x}{x^2 \pm a^2} dx = \frac{1}{2} \ln\left| x^2 \pm a^2 \right| + C$
4. $\int \frac{x}{(x \pm a)^2} dx = \int \frac{x + a - a}{(x \pm a)^2} dx = \int \frac{x \pm a}{(x \pm a)^2} dx \mp a \int \frac{dx}{(x \pm a)^2} = \ln|x \pm a| \pm \frac{a}{x \pm a} + C$

<u>Example 1.</u> Compute integrals

a) $\int \frac{dx}{2x^2 + 3}$,

b) $\int \frac{dx}{x^2 + 3x + 4}$,

c) $\int \frac{dx}{3x^2 - 15}$,

d) $\int \frac{dx}{x^2 - 10x + 16}$,

e) $\int \frac{x}{x^2 + x + 1} dx$,

f) $\int \frac{2x + 3}{x^2 + 6x + 9} dx$,

g) $\int \frac{x^4}{x^2 + 1} dx$,

h) $\int \frac{2x + 5}{x^2 + 4x + 5} dx$

Remark. If $x^2+px+q=(x-x_1)(x-x_2)$, where x_1 and x_2 are two different real numbers, then there exist real constants A and B such that $\dfrac{ax+b}{x^2+px+q}=\dfrac{A}{x-x_1}+\dfrac{B}{x-x_2}$, what is useful for integration. Unknown constants A and B are found by "the method of indefinite coefficients".

<u>Example 2</u>. By means of decomposition of the integrand calculate $\displaystyle\int\frac{12x+2}{x^2+5x-6}dx$

By the substitution method it is possible to transform integrals of some simple irrational functions to the integrals of rational functions.

<u>Example 3.</u> With the aid of change of variable calculate integrals:

a) $\displaystyle\int\frac{x}{\sqrt[3]{x+1}}dx \quad (x+1=t^3)$, b) $\displaystyle\int\sqrt{\frac{1-x}{1+x}}\cdot\frac{dx}{(1+x)^2} \quad \left(\frac{1-x}{1+x}=t^2\right)$,

c) $\displaystyle\int\frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx \quad (x=t^4)$, d) $\displaystyle\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}} \quad (x=t^6)$,

e) $\displaystyle\int\frac{\sqrt[3]{x-1}+x}{x-1}dx \quad (x-1=t^3)$

Frequently occuring integrals of irrational functions are: $\displaystyle\int\frac{dx}{\sqrt{ax^2+bx+c}}$, where $a\neq 0$ and ax^2+bx+c is possitive on an interval. We can exclude the case, that the polynomial ax^2+bx+c has a double root.

Taking the factor $\dfrac{1}{\sqrt{a}}$ (if $a>0$) or $\dfrac{1}{\sqrt{-a}}$ (if $a<0$) we reduce the integral to the form $\displaystyle\int\frac{dx}{\sqrt{x^2+px+q}}$ or $\displaystyle\int\frac{dx}{\sqrt{-x^2+px+q}}$, leading (by means of substitution) to the integrals:

$\displaystyle\int\frac{dx}{\sqrt{x^2\pm k^2}}=\ln\left|x+\sqrt{x^2\pm k^2}\right|+C$ or $\displaystyle\int\frac{dx}{\sqrt{k^2-x^2}}=\arcsin\frac{x}{k}+C$, respectively.

<u>Example 4.</u> Calculate integrals:

a) $\displaystyle\int\frac{dx}{\sqrt{x^2-6x+13}}$, b) $\displaystyle\int\frac{dx}{\sqrt{3x^2+5x+4}}$,

c) $\displaystyle\int\frac{dx}{\sqrt{3-2x-x^2}}$, d) $\displaystyle\int\frac{dx}{\sqrt{x-2x^2}}$

Integrating Trigonometric Functions

Given an integral $\int R(\sin x, \cos x)dx$, i.e. the integrand is a rational function in terms of $\sin x$ and $\cos x$. By the substitution $t = \tan\frac{x}{2}$ the integral is reduced to an integral of a rational function. If $t = \tan\frac{x}{2}$, then $x = 2\arctan t$, $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

Example 5. Compute integrals:

a) $\int\frac{dx}{\sin x}$, b) $\int\frac{dx}{3+2\cos x}$,

c) $\int\frac{dx}{\cos x}$, d) $\int\frac{dx}{1-\sin x}$

Generally, this method is very convenient for computing integrals of the form $\int\frac{dx}{a\cos x + b\sin x + c}$

If the integrand can be reduced to the form $f(\sin x)\cos x$ or $f(\cos x)\sin x$, where f is a simply integrable function, then it is advantageous to put $t = \sin x$ or $t = \cos x$, respectively.

In contrast to differentiation, integration of an elementary function not always leads to an elementary function. It can be proved that there exist elementary functions whose integrals are inexpressible in terms of elementary functions. For instance, the integrals

$$\int\frac{dx}{\sqrt{1+x^3}},\ \int\frac{e^x}{x}dx,\ \int\frac{\sin x}{x}dx,\ \int\frac{\cos x}{x}dx,\ \int\frac{dx}{\ln x} \text{ and } \int e^{-x^2}dx$$

cannot be represented by any elementary functions. We must distinguish between the question of existence of a desired antiderivative and the possibility of expressing it with the aid of elementary functions.

The integrals written above exist, but the class of all elementary functions which we use is insufficient for expressing these integrals.
To find these integrals it is necessary to extend the class of functions, we use. This is exactly what is done in mathematical analysis.
The nonelementary functions determined by the most important integrals inexpressible in terms of elementary functions are thoroughly investigated and tabulated.