Rules of Differentiation,
Derivatives of Elementary Functions, Differential.

Basic Rules of Differentiation.

If functions $f(x)$ and $g(x)$ are differentiable on a set *M*, then functions $c \cdot f(x)$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$ (if $g(x) \neq 0, \forall x \in M$) are differentiable on *M* as well, and:

1. $[c \cdot f(x)]' = c \cdot f'(x), c \in R$
2. $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
3. $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

These rules follow directly from the definition of derivative at a point.

Derivative of a Composite Function (The Chain Rule). If a function $\varphi(x)$ has derivative at a point x_0 and a function $f(u)$ has derivative at the point $u_0 = \varphi(x_0)$, then the composite function $F(x) = f(\varphi(x))$ has also derivative at the point x_0 and it holds: $F'(x_0) = f'(u_0) \cdot \varphi'(x_0)$, $u_0 = \varphi(x_0)$

Another form: $\left[f(\varphi(x)) \right]' = f'(\varphi(x)) \cdot \varphi'(x)$, $\varphi(x) = u$ (on a set).

Derivatives of Basic Elementary Functions. From the definition of derivative and the Rules of differentiation are derived following formulas:

1. $(c)' = 0, \quad c \in R, \quad x \in R$
2. $(x^a)' = ax^{a-1}, \quad a \in R, \quad x \in (0, \infty)$
3. $(a^x)' = a^x \cdot \ln a, \quad a > 0, a \neq 1, \quad x \in R \quad \left((e^x)' = e^x\right)$
4. $(\log_a x)' = \frac{1}{x \cdot \ln a}, \quad a > 0, a \neq 1, \quad x \in (0, \infty) \quad \left((\ln x)' = \frac{1}{x}\right)$
5. $(\sin x)' = \cos x, \quad x \in R$
6. $(\cos x)' = -\sin x, \quad x \in R$
7. $(\tan x)' = \frac{1}{\cos^2 x}, \ x \neq (2k-1) \cdot \frac{\pi}{2}, \ k \in Z$
8. $(\cot x)' = -\frac{1}{\sin^2 x}, \ x \neq k \cdot \pi, \ k \in Z$

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \; x \in (-1,1)$

10. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1)$

11. $(\arctan x)' = \dfrac{1}{(1+x^2)}, \; x \in R$

12. $(arc\cot gx)' = \dfrac{-1}{(1+x^2)}, \; x \in R$

<u>Example 1.</u> Differentiate functions: $f_1 : y = \dfrac{\ln x}{x^2}$, $f_2 : y = 2^x \cdot \arccos 3x$, $f_3 : y = 5 \cdot \arctan^2 x$, $f_4 : y = \sqrt{1-x} \cdot \ln(\sin x)$

Logarithmic Differentiation

If $F(x) = [f(x)]^{g(x)}$ then $F'(x) = [f(x)]^{g(x)} \cdot \left(g'(x) \cdot \ln(f(x)) + g(x) \cdot \dfrac{f'(x)}{f(x)} \right)$ for $f(x) > 0$ **.)**

<u>Example 2</u>. Find derivatives: $f_1 : y = x^{\arctan x}$, $f_2 : y = (\sin x)^{\ln x}$, $f_3 : y = (x^2 + 2)^{\cos 3x}$

Derivatives of Higher Orders

If a function $f(x)$ is differentiable on a set M and if its derivative $f'(x)$ has derivative at each point $x \in M$, then this derivative is called the **second derivative** of $f(x)$ on M and it is denoted $f''(x)$, or $\dfrac{d^2 f}{dx^2}$.

Analogously the third derivative is defined, and so on. In general:
If for all points $x \in M$ the function $f^{(n-1)}(x)$ (the derivative of (n-1)-th order) is differentiable then its derivative is called the ***n*-th derivative** , or derivative of n-th order of *f:*
It means that $f^{(n)}(x) = [f^{(n-1)}(x)]'$, for n=2,3,4,…

Another notation of the n-th derivative is $\dfrac{d^n f}{dx^n}$.

<u>Example 3.</u> Find $f'''(x)$, if $f : y = arctg\dfrac{1}{x}$ and $f^{(10)}(x)$, if $f : y = e^{5x}$.

<u>Example 4.</u> Find the formula for the n-th derivative of $f : y = \dfrac{1}{x}$

Differential and Its Geometric Meaning

Suppose a function $f(x)$ is defined in $N_\varepsilon(x_0)$ and differentiable at x_0. The difference $\Delta f = f(x) - f(x_0)$ is called the increment of the function, corresponding to the increment of independent variable $\Delta x = x - x_0$. The expression $f'(x_0)(x - x_0)$ is called the differential of $f(x)$ at x_0 and it is denoted df_{x_0} : $df_{x_0} = f'(x_0)(x - x_0)$

The differential df_{x_0} of a function $f(x)$ at a point x_0 is equal to the increment of the y-coordinate of the tangent line drawn to the $G(f)$ at the point $\left[x_0,\ f(x_0)\right]$. If the difference $x - x_0$ approaches 0, then $\Delta f \doteq df$, thus $f(x) - f(x_0) \doteq f'(x_0)(x - x_0)$

For the function $f: y = x$, we have $df = dx = \Delta x$, that is why the differential at an arbitrary point is denoted : $df = f'(x) \cdot dx$.

<u>Example 5.</u> Find the differential of $f: y = \arctan x$, at $x_0 = 2$ and its value at $x = 3$.

<u>Example 6.</u> By means of differential calculate approximately $\sqrt{408}$.

<u>Example 7.</u> The radius of a circle is to be increased from the initial value of $r_0 = 10$ by an amount $dr = 0.1$. Estimate the corresponding increase in the circle's area $A = \pi r^2$ by calculating dA. Compare dA with the true change ΔA.

<u>Example 8.</u> An edge of a cube is measured as 6 in. with a possible error of ± 0.05 in. The volume of the cube is to be calculated from this measurement. Estimate the error that would occur in the volume calculation.

<u>Example 9.</u> Differentiate:
$y = x^2 \cdot \sqrt{x} - 2^{-x} \cdot \ln x$, $y = \arccos 2x \cdot \log_4\left(x^2 - x\right)$, $y = x^x \cdot \tan(\sin 3x)$

<u>Example 10</u>. Write equation for the tangent line to the graph of $f: y = 4x + \dfrac{3}{x}$, parallel to the stright line $p: 11x - 3y + 2 = 0$

<u>Example 11.</u> By means of differential calculate aproximately the following values:
arcsin 0.2, tan 46^{o}, $2^{1.002}$, arctan 1.1.
Compare calculated values with the values, found by means of calculator.