Properties of Continuous Functions
Asymptotes of Graphs, Derivatives

Properties of Continuous Functions on a closed Interval.

Functions, continuous on a closed interval $\langle a, b \rangle$ possess a number of important properties. Here we present 3 of them:

1) A function, continuous on a closed interval, is bounded on this interval
2) A function f continuous on a closed interval $\langle a, b \rangle$, assumes the greates value (maximum) and the least value (minimum) in this interval, e.a.: $\exists c_1, c_2 \in \langle a, b \rangle$, such that: $f(c_1) \leq f(x) \leq f(c_2), \forall x \in \langle a, b \rangle$
3) If a function f is continuous on a closed interval $\langle a, b \rangle$ and $f(a) \cdot f(b) < 0$, then there exists $c \in (a, b)$, such that $f(c) = 0$.

Corollary (Consequence). If a function f is continuous on a closed interval $\langle a, b \rangle$, then the set (image of $\langle a, b \rangle$): $f(\langle a, b \rangle) = \{f(x) : x \in \langle a, b \rangle\}$ is again a closed interval or a one-point set.

<u>Example 1.</u> Show, that $f : y = \sqrt{1 - x^2}$ is bounded and that it assumes maximum and minimum (on $D(f)$).

<u>Example 2</u>. Show, that equations:

1. $x^3 + x + 1 = 0$ and
2. $e^x + x = 0$

have at least one root in $\langle -1, 0 \rangle$

<u>Example 3</u>. Find minimum and maximum for $f_1 : y = \cos x$ on $\langle 0, 4\pi \rangle$ and $f_2 : y = x^2 - 2x$ on $\langle 0, 4 \rangle$.

<u>Example 4.</u> Find $f(\langle 0, 3 \rangle)$ for $f(x) = 5$, $f(\langle 0, 2 \rangle)$ for $f(x) = x^2 - 2x$, $f(\langle 0, 2\pi \rangle)$ for $f(x) = 1 + \cos\frac{x}{2}$ and $f(\langle 1, e \rangle)$ for $f(x) = \ln x$.

Asymptotes of Graphs

A straight line is an asymptote of the graph of a function, if the distance from the variable point M of the graph to this line approaches zero, as the point M recedes to infinity (asymptotes are tangents at infinity).

One should distinguish between vertical (without the slope) and inclined (with the slope) asymptotes.

A straight line: $x = a$ is said to be an **asymptote of** $G(f)$ **without the slope** (parallel to the axis O_y), if at least one of the following equalities is fulfilled: $\lim_{x \to a^+} f(x) = +\infty$ (or $-\infty$), $\lim_{x \to a^-} f(x) = +\infty$ (or $-\infty$).

A straight line: $y = kx + b$ is said to be an **asymptote of** $G(f)$ **with the slope** (not parallel to O_y) as x approaches infinity, if

$k = \lim_{x \to \infty} \frac{f(x)}{x}$ and $b = \lim_{x \to \infty} (f(x) - kx)$ (k and b are numbers).

Analogically as x approaches $-\infty$.

In particular, if the function f tends to a finite limit as $x \to \infty$, that is $\lim_{x \to \infty} f(x) = b$, then obviously $k = 0$ and $G(f)$ has a horizontal asymptote (regarded as a special case of the inclined asymptote) parallel to O_x, namely $y = b$. Similarly if $x \to -\infty$.

The asymptotic behaviour of a function may be of different character when x becomes positively or negatively infinite, and therefore cases $x \to +\infty$ and $x \to -\infty$ should be treated separately.

If k and b for $x \to +\infty$ and $x \to -\infty$ coinside, then both asymptote form a common straight line.

<u>Example 5</u>. Find all asymptotes for the function $f : y = \frac{x^2 + 3x + 5}{x + 1}$

<u>Example 6</u>. Show, that straight lines $y = x$ and $y = -x$ are inclined asymptotes as $x \to +\infty$ and $x \to -\infty$, resp., for $f : y = \sqrt{x^2 - 9}$.

Derivative of a function at a point (derivative as a number)

If a function $f(x)$ is defined in $N_\varepsilon(x_0)$ and if there exists $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ (the proper limit), then this limit is said to be a **derivative of f at** x_0, and denoted $f'(x_0)$. Therefore

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \left(= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \Delta x = x - x_0 \right)$$

If $f'(x_0)$ exists, the function f is called **differentiable at** x_0. $f'(x_0)$ is a real number!

<u>Example 7</u>. Show, that the function $f_1 : y = x^2$ is differentiable at $x_0 = 3$ and $f_2 : y = |x|$ is not differentiable at $x_0 = 0$.

If $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = +\infty \quad (or\ -\infty)$, it is said, that f has at x_0 an **improper derivative** (but f is not differentiable at x_0).

Example 8. Show, that the function $f: y = \sqrt[3]{x}$ has an improper derivative at $x_0 = 0$.

Derivative of a function on a set (derivative as a function). Let's denote $M = \{x \in D(f) : \exists\ f'(x)\}$. We can define on M a new function
$f': y = f'(x), \forall x \in M$. This function is called **derivative of f** and denoted $f'(x)$.
For example: If $f: y = x^2 \Rightarrow f': y = 2x$ on $M = (-\infty, \infty)$.
We also write $(x^2)' = 2x$, or: $\frac{df}{dx} = 2x$, if $f(x) = x^2$.

Necessary condition of differentiability: If a function $f(x)$ is differentiable at a point x_0, it is continuous at this point: $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0) \right) = f(x_0)$

Continuity is not sufficient condition of differentiability. The function $f: y = |x|$ is continuous at $x_0 = 0$, but not differentiable.

Geometrical Meaning of the Derivative.

If the value of the derivative of a function $f(x)$ at a point x_0 is $f'(x_0)$, then the straight line: $y - f(x_0) = f'(x_0)(x - x_0)$ is the tangent to the graph of $f(x)$ at the point $[x_0, f(x_0)]$. Hence $f'(x_0)$ is the slope of the tangent to $G(f)$ at $[x_0, f(x_0)]$.

Example 9. Find the tangent to the graph of $f: y = x^2$, if $x_0 = 3$.

Physical Meaning of the Derivative.

If a point moves along a straight line and its law of motion is $s = f(t)$ (*t*-time), then the ratio $\frac{f(t) - f(t_0)}{t - t_0}$ is an average velocity of the motion, corresponding to the time interval $\Delta t = t - t_0$. Then $v(t_0) = f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$ is called velocity of the rectilinear motion $s = f(t)$ at the given moment $t = t_0$.

Example 10. Find the velocity of uniformly accelerated motion $s = \frac{1}{2} g \cdot t^2$ (law of free fall) at $t_0 = 2$ and show, that the velocity is directly proportional to the time of motion.