Limits and Continuity

Neighbourhoods.

Let $\varepsilon > 0$ and $a \in R$, then $N_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$ is called the ε-neighbourhood of the number a (the point a). $N_\varepsilon^+ = (a, a + \varepsilon)$ is called the ε-right-hand neighbourhood and $N_\varepsilon^- = (a - \varepsilon, a)$ the ε-left-hand neighbourhood of the number a.

Example 1. Consider $f: y = 2x + 1$, $D(f) = R$, $x = 2 \in D(f)$, $f(2) = 5$. Let $\varepsilon > 0$ and $f(x) \in N_\varepsilon(5)$,e.a.:

$5 - \varepsilon < 2x + 1 < 5 + \varepsilon \Leftrightarrow 4 - \varepsilon < 2x < 4 + \varepsilon \Leftrightarrow 2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2} \Leftrightarrow x \in N_\delta(2)$ for $\delta = \frac{\varepsilon}{2}$. It follows, that if x tends to 2, $f(x)$ tends to 5.

Limit of a function $f(x)$ at a point *a* (proper).

Let $f(x)$ be defined in a neighbourhood of a point a, $x \neq a$. A number b is said to be a limit of the function f at the point a, if for any $N_\varepsilon(b)$ there exists $N_\delta(a)$ such that for $\forall x \in N_\delta(a)$, $x \neq a$ is $f(x) \in N_\varepsilon(b)$.

It is written: $\lim\limits_{x \to a} f(x) = b$

Hence: $\lim\limits_{x \to a} f(x) = b \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta, x \neq a \Rightarrow |f(x) - b| < \varepsilon$

In Example 1: $\lim\limits_{x \to 2} f(x) = 5$

Example 2. Show, that $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$ (By means of definition)

Example 3. Show, that $\lim\limits_{x \to a} x = a$, $\forall a \in R$.

Remark. The function $f: y = \sqrt{x + 2}$ is defined on $D(f) = \langle -2, \infty)|$. It follows, that there is no $\delta > 0$, such that $N_\delta(-2) \subset D(f)$. Therefore $\lim\limits_{x \to -2} \sqrt{x + 2}$ cannot exist.

If in the definition of limit we replace $N_\delta(a)$ by $N_\delta^+(a)$ or $N_\delta^-(a)$, we obtain definitions of one-sided limits:

$\lim\limits_{x \to a^+} f(x) = b \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : x \in N_\delta^+(a) \Rightarrow f(x) \in N_\varepsilon(b)$ (b is called limit on the right of $f(x)$ at a) and $\lim\limits_{x \to a^-} f(x) = b \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : x \in N_\delta^-(a) \Rightarrow f(x) \in N_\varepsilon(b)$ (b is called limit on the left of $f(x)$ at a). It can be shown, that $\lim\limits_{x \to -2^+} \sqrt{x + 2} = 0$.

Example 4. Show, that $\lim\limits_{x\to 0}\frac{|x|}{x}$ doesn't exist.

Basic properties of limits.

1. Any function at any point has at most one limit.
2. If there exists a limit of a function f at a point a, then the function f is bounded at a neighbourhood of the point a.
3. If $f(x)=c,\ c\in R,\ D(f)=R \Rightarrow \lim\limits_{x\to a} f(x)=c,\ \forall a\in R$
4. Let $\lim\limits_{x\to a} f(x)=A$ and $\lim\limits_{x\to a} g(x)=B$ Then:

a) $\lim\limits_{x\to a}\left(f(x)\pm g(x)\right)=\lim\limits_{x\to a} f(x)\pm \lim\limits_{x\to a} g(x)=A\pm B$

b) $\lim\limits_{x\to a}\left(f(x)\cdot g(x)\right)=\lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)=A\cdot B$

c) $\lim\limits_{x\to a} c\cdot f(x)=c\cdot \lim\limits_{x\to a} f(x)=c\cdot A$

d) $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{A}{B}$, if $B\neq 0$ and $g(x)\neq 0$ on an $N_\varepsilon(a)$, for $x\neq a$.

e) $\lim\limits_{x\to a}\left[f(x)\right]^k=\left[\lim\limits_{x\to a} f(x)\right]^k=A^k,\ k\in N$

5. If $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} h(x)=b$ and if there exists $N_\varepsilon(a)$ such that: $\forall x\in N_\varepsilon(a),\ x\neq a:\ f(x)\leq g(x)\leq h(x)$, then $\lim\limits_{x\to a} g(x)=b$. All these properties are valid also for one-sided limits

Example 5. Compute: $\lim\limits_{x\to 3}\left(x^2+4\right)$, $\lim\limits_{x\to -1}\frac{2x^5}{x^2+1}$, $\lim\limits_{x\to 2}\frac{x^2-4}{x^3-8}$

Improper limits and limits at improper points.

Example 6. Consider $f: y=\frac{1}{x^2}\quad D(f)=R-\{0\}$

Since $f(x)$ is unbounded in any neighbourhood $N_\varepsilon(0)$, $\lim\limits_{x\to a} f(x)$ doesn't exist. But $f(x)$ has the following property: For each $K>0:\ f(x)>K \Leftrightarrow x\in\left(-\frac{1}{\sqrt{K}},\frac{1}{\sqrt{K}}\right),\ x\neq 0$.

For example: $\frac{1}{x^2}>10^6 \Leftrightarrow 0<|x|<10^{-3}$

It follows: if x tends to 0, $f(x)$ tends to ∞

Improper limit of a function *f(x)* at a point *a*.

Let $f(x)$ be defined in a neighbouhood of a point *a*, $x \neq a$. It is said, that *f* has an improper limit $+\infty$ (or $-\infty$) at the point a and it is written $\lim\limits_{x \to a} f(x) = \infty \ (or - \infty) \Leftrightarrow \forall K > 0 : \exists N_\delta(a) : x \in N_\delta(a), x \neq a \Rightarrow f(x) > K \ (or\ f(x) < -K)$

One-sided improper limits at *a* are defined analogously.

Example 7. By means of definition find: $\lim\limits_{x \to 0^+} \dfrac{1}{x}$ and $\lim\limits_{x \to 0^-} \dfrac{1}{x}$

Let's introduce two more properties of limits:

6. If $\lim\limits_{x \to a} f(x) = b \neq 0$ and $\lim\limits_{x \to a} g(x) = 0$ and if there exists $N_\varepsilon(a)$ such that: $\forall x \in N_\varepsilon(a), x \neq a : \dfrac{f(x)}{g(x)} > 0 \left(or\ \dfrac{f(x)}{g(x)} < 0 \right)$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \infty \left(or\ \dfrac{f(x)}{g(x)} = -\infty \right)$.
7. If $f(x)$ is bounded in an $N_\varepsilon(a)$ and $\lim\limits_{x \to a} g(x) = \pm\infty$ then $\lim\limits_{x \to a} (f(x) + g(x)) = \pm\infty$, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$. These properties are valid for one-sided limits as well.

Example 8. Compute: $\lim\limits_{x \to 2} \dfrac{1 + x}{(x - 2)^2}$, $\lim\limits_{x \to 0} \left(5 + \dfrac{1}{x^2} \right)$

Limits and improper limits at $+\infty$ (or $-\infty$) .

Let $f(x)$ be defined on an interval (a, ∞).
Then:

$$\lim_{x \to \infty} f(x) = b (b \in R) \Leftrightarrow \forall N_\varepsilon(b) : \exists A > 0 : x > A \Rightarrow f(x) \in N_\varepsilon(b)$$

$$\lim_{x \to \infty} f(x) = \infty \ (or - \infty) \Leftrightarrow \forall K > 0 : \exists A > 0 : x > A \Rightarrow f(x) > K \ (or\ f(x) < -K)$$

If $f(x)$ is defined on an interval $(-\infty, a)$, limits at $-\infty$ are defined similarly: $\lim\limits_{x \to -\infty} f(x) = b$ and $\lim\limits_{x \to -\infty} f(x) = \infty \ (or - \infty)$.

Properties 1. - 7. hold for limits at improper points $\pm\infty$ too.

Example 9. Compute $\lim\limits_{x \to \infty} \dfrac{2x^2 + 3x}{x^2 + 4}$, $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x}{2 - x^3}$, $\lim\limits_{x \to \infty} \dfrac{2 + x^3}{1 - x^2}$

Continuity of a function *f(x)* at a point *a*.

It is said that a function $f(x)$ is continuous at a point *a* if $\lim_{x \to a} f(x) = f(a)$. It means:

1. $f(x)$ is defined at a $(a \in D(f))$,
2. there exists $\lim_{x \to a} f(x)$,
3. this limit is equal to $f(a)$.

It is said, that f at *a* is continuous on the right (or on the left), if

$$\lim_{x \to a^+} f(x) = f(a) \text{ (or } \lim_{x \to a^-} f(x) = f(a)\text{).}$$

Remark. A function $f(x)$ is said to be continuous on an interval $\langle a, b \rangle$ if it is continuous at each $x \in (a, b)$ and moreover, if it is continuous at *a* on the right and at *b* on the left.

From properties 4a) – 4e) it follows:
If $f(x)$ and $g(x)$ are continuous at a point *a*, then at this point are continuous also the functions:

1. $f(x) \pm g(x)$,
2. $c \cdot f(x), c \in R$,
3. $f(x) \cdot g(x)$,
4. $\frac{f(x)}{g(x)}$, if $g(a) \neq 0$ and
5. $[f(x)]^k$, $k \in N$

All elementary function are continuous at each point of their domains of definition.

Points of discontinuity.

If a function *f* is not continuous at a point *a*, the point *a* is called a point of discontinuity of *f*.
There are 3 possibilities for a point *a*, to be the point of discontinuity:

1. $f(x)$ has no limit at a
2. $a \notin D(f)$
3. $\lim_{x \to a} f(x) \neq f(a)$.