Functions of a real Variable

Basic Notions

Let M be a nonempty set of real numbers: $M \neq \varnothing$, $M \subset \boldsymbol{R}$. A rule f, that assignes to each element (real number) $x \in M$ just one element (real number) $y \in \boldsymbol{R}$ is called a real function of a real variable, briefly a function f. The set M is called the domain of definition f and it is denoted by $D(f)$. $(M = D(f))$

The number $y = f(x)$ is the value of f at x. The set of real numbers which are values of f, $R(f) = \{y : \exists x \in D(f) : y = f(x)\}$ is called the range of f. y is said to be dependent and x is independent variable (or argument).

The set $G(f) = \{[x, y] : x \in D(f), y = f(x)\} \subset \boldsymbol{R} \times \boldsymbol{R}$ is called the graph of f. A set of points in plane is graph of a function, if each straight line parallel to O_y (y-axis) has at most one common point with it. $D(f)$ is orthogonal projection of $G(f)$ onto O_x (x-axis) and $R(f)$ is orthogonal projection of $G(f)$ onto O_y (y-axis).

Remark 1. If a function is given by an analytic formula, without specifying $D(f)$, then we are interested in those real x, for which the formula makes a sense. The set of all those x is then accepted as $D(f)$ of the given function and it is said to be the natural (maximal) domain of definition.

Example 1.

Find D(f), R(f) and sketch G(f) of functions

a) $f: y=1-x^2$, b) $f: y=2+\sqrt{x}$, c) $f: y=|x-1|$

Operations on Functions

Let f and g be two functions with domains $D(f)$ and $D(g)$.

- Functions f and g are equal one another, if $D(f)=D(g)$ and if for each x from the domain $f(x)=g(x)$.
- Function F, defined on $D(F)= D(f) \cap D(g)$ is called sum (difference,product,quotient), of functions f and g and denoted $f+g$ ($f-g$, $f.g$, $\frac{f}{g}$), if for each $x \in D(F)$:
- $F(x)=f(x)+g(x)$, ($F(x)=f(x)-g(x)$, $F(x)=f(x).g(x)$, $F(x)=\frac{f(x)}{g(x)}$) .
- Apparently, points at which $g(x)=0$ must be excluded from $D(f) \cap D(g)$, to obtain the domain of the quotient $\frac{f}{g}$.

Example 2.

If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$, give the domains of $f+g$ and $\frac{f}{g}$.

Composite Functions.

Suppose that the outputs of a function *g* with the domain $D(g)$ can be used as inputs of a function *f* with the domain $D(f)$. We can then hook *f* and *g* together to form a new function *F*, whose inputs are inputs of *f* and whose outputs are the numbers $f(g(x))$.
More precisely:
Put $D(F)=\{x \in D(g): g(x) \in D(f)\}$ The function *F*, defined on $D(F)$ so that
$F(x)= f(g(x)), \quad \forall x \in D(F)$, is called a **composite** function, the function $f(u)$ is its major (outside) component and the function $u = g(x)$ its minor (inside) component.

Example 3.

If we have two functions *f: y*=sin *x* and *g: y=x-* π , write formulas for composites *f(g(x))* and *g(f(x)).* Then find their values at 0.

Example 4.

Let f(x) =ln(2- x) and g(x) = $\sqrt{x+3}$. Find the formula and the domain of definition for the composite $F(x)= f(g(x))$.

Some Special Classes of Functions

Bounded functions. A function *f* is called bounded (bounded above, bounded below), if there exists a real number *K* such that *|f(x)|<K* (*|f(x)<K, f(x)>K*), for each *x* from *D(f).*
It means, that a function is bounded (bounded above, bounded below), if its range R(f) is a bounded (bounded above, bounded below) set of real numbers.

Example 5.

Let $f(x)=\dfrac{1}{x^2}, g(x)=-\sqrt{x+1}$. Show, that the function *f* is bounded below and the function *g* is bounded above.

Monotone Functions. We distinguish 4 types of monotone functions: Incresing, decreasing, non-decreasing and non-incresing functions.

A function *f* is called incresing (decresing, non-decreasing, non-increasing), if for each couple of points:
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ $(f(x_1) > f(x_2), f(x_1) \le f(x_2), f(x_1) \ge f(x_2))$

It is clear, that any increasing function is non-decreasing and any decreasing function is non-increasing, but opposite is not true. Constant functions represent the only possible type of functions, which are non-decreasing and non-increasing simultaneously.

Incresing and decresing functions are said to be **strictly monotone**.

Remark 2. In the sence of the foregoing definition for instance the function *f: y=x*^{2} is not monotone, because none of required conditions is fulfilled for each couple of points in its domain of definition. Nevertheless, if we restrict ourselves to the set of non-negative numbers, it is easy to see, that *f* increasis there.

Similarly, *f* decreasis on the set of non-positive numbers.

Example 6.

Show, that the function *f(x)* = 3 - $\dfrac{1}{\sqrt{x}}$ is strictly monotone.

Periodic Functions. A function *f* is called periodic, if there exists a positive number *p* such that if $x \in D(f) \Rightarrow x \pm p \in D(f)$ and $f(x+p) = f(x)$,
for each $x \in D(f)$.
The number *p* is said to be a **period** of the function *f*.
Remark 3. All functional values of a periodic function repeat themselves infinitely many times, in other words also the part of graph on any interval of the length *p* is repeated infinitely many and whole graph consists of copies of it.
Remark 4. Trigonometric functions are the most frequently appearing type of periodic functions. The functions $\sin x$ and $\cos x$ have the least period $p=2\pi$, $\tan x$ and $\cot x$ have the least period $p=\pi$.
Even and Odd Functions. A function f is called even (odd), if

$$x \in D(f) \Rightarrow -x \in D(f) \text{ and } f(-x) = f(x) \quad (f(-x) = -f(x)),$$

for each $x \in D(f)$.
Remark 5. It can be easily proved, that graphs of even functions are symmetrical with respect to O_y and those of odd functions are symmetrical with respect to origin of coordinate system, the point [0,0].
<u>Example 7.</u>
Determine whether the functions $f(x)=x^3 - 2x$ and $g(x) = 1- 3x^2$ are even or odd functions.

One-to-one Functions Let f be a function with the domain $D(f)$. If for $x_1, x_2 \in D(f)$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, the function f is said to be one-to-one.
A function f is one-to-one if each straight line parallel to O_x has at most one common point with $G(f)$.

Inverse Functions Let f be a one-to-one function with the domain $D(f)$ and the range $R(f)$ and let a function f^{-1} be defined on $R(f)$ as follows: For each $y_\circ \in R(f)$: $f^{-1}(y_\circ) = x_\circ \in D(f)$, if $f(x_\circ) = y_\circ$. Then the function f^{-1} is called the inverse function of f.
Obviously: $D(f^{-1}) = R(f)$, $R(f^{-1}) = D(f)$, $(f^{-1})^{-1} = f$
Remark. Any inverse function is again one-to-one. Graphs of two mutually inverse functions *f* and f^{-1}, e.a. $G(f)$ and $G(f^{-1})$ are symmetric (one another) with respect to the straight line $y = x$.
Trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x \ (tgx)$, $y = \cot x$ are periodic, it means that they assume each value from their ranges infinitely many times. It follows that they are not one-to-one and they have no inverse functions.
But if instead of these functions, defined on their natural domains we consider them only on appropriate parts of domains, namely $y = \sin x \Big/ \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$, $y = \cos x / \langle 0, \pi \rangle$,

$y = \tan x \Big/ \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ and $y = \cot x / (0, \pi)$ we obtain one-to-one functions.

Cyclometric functions

The function $f_1 : y = \sin x / \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$ is increasing, thus there exists its inverse function, called $\arcsin x$, $f_1^{-1} : y = \arcsin x$

The function $f_2 : y = \cos x / \langle 0, \pi \rangle$ is decreasing, thus there exists its inverse function, called $\arccos x$, $f_2^{-1} : y = \arccos x$

They have following basic properties:
1. $D\left(f_1^{-1}\right) = D\left(f_2^{-1}\right) = \langle -1, 1 \rangle$
2. $R\left(f_1^{-1}\right) = \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$, $R\left(f_2^{-1}\right) = \langle 0, \pi \rangle$
3. For $\forall x \in \langle -1, 1 \rangle$:

$\arcsin x = y \Leftrightarrow \sin y = x,\ y \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$, $\arccos x = y \Leftrightarrow \cos y = x,\ y \in \langle 0, \pi \rangle$

The function $g_1 : y = \tan x / \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ is increasing, thus there exists its inverse function, called arctangent x, $g_1^{-1} : y = \arctan x$

The function $g_2 : y = \cot x / (0, \pi)$ is decreasing, thus there exists its inverse function, called arccotangent x, $g_2^{-1} : y = arc \cot x$.

They have following basic properties:
1. $D\left(g_1^{-1}\right) = D\left(g_2^{-1}\right) = (-\infty, \infty)$
2. $R\left(g_1^{-1}\right) = \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, $R\left(g_2^{-1}\right) = (0, \pi)$
3. For $\forall x \in (-\infty, \infty) \quad (x \in R)$:

$arctgx = y \Leftrightarrow tgy = x,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, $arc \cot gx = y \Leftrightarrow \cot gy = x,\ y \in (0, \pi)$

All four functions f_1^{-1}, f_2^{-1}, g_1^{-1}, g_2^{-1} are bounded and strictly monotone, f_1^{-1} and g_1^{-1} are increasing and f_2^{-1} and g_2^{-1} are decreasing.

Elementary Functions

Functions: constants, $x^r (r \in R)$, a^x, $\log_a x$, $\sin x$, $\cos x$, tgx, $\cot gx$, $\arcsin x$, $\arccos x$, $arctgx$ and $arc \cot x$ are usually called basic elementary functions. Functions, constructed of basic elementary functions by means of a finite number of arithmetic operations: addition, subtraction, multiplication and division and operations of forming composite functions are called elementary functions.