Matrices, Determinants, Systems of Linear Equations

Matrices

Let *m* and *n* be natural numbers. A rectangular array of real numbers, arranged to *m* rows and *n* columns, enclosed in parentheses, written in the way

$$\begin{pmatrix} a_{11}, & a_{12},\dots & a_{1n} \\ a_{21}, & a_{22},\dots & a_{2n} \\ a_{m1}, & a_{m2},\dots & a_{mn} \end{pmatrix}$$

is called a matrix of the type $m \times n$.

It is denoted $A = (a_{ij})$, $i = 1,2,\dots m$, $j = 1,2,\dots n$ Numbers a_{ij} are elements of *A*. If $m = n$, A is called a square matrix.
The set of elements $(a_{11}, a_{22}, a_{33},\dots a_{kk})$, where $k = \min(m, n)$ is called diagonal (main or major) of A.

Operations on Matrices

Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ be matrices of the same type $m \times n$. Then

1. A = B if $a_{ij} = b_{ij}$, $\forall\, i = 1,2,\dots m,\ j = 1,2,\dots n$
2. Matrix addition: Matrix $C = (c_{ij})$, such that $C = a_{ij} + b_{ij}$, $\forall i, j$ is called sum of A and B, C = A + B
3. Multiplying Matrices and Numbers
   Let *k* be a number. Matrix $B = (b_{ij})$, such that $b_{ij} = k \cdot a_{ij}$, $\forall\, i, j$ is called product of *k* and A, B = k.A

4. Multiplication of matrices. Let $A = (a_{ij})$ be a matrix m × n and $B = (b_{ij})$ a matrix n × p. Matrix $C = (c_{ij})$, of the type m × p, such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{mj}$ $\forall i = 1,2,\dots m,\ j = 1,2,\dots p$ is called product of A and B, C = A.B
Multiplication of matrices, in general, is not commutative $A \cdot B \neq B \cdot A$

Determinants

Consider a system of linear equations in two unknowns (variables):

$$\begin{aligned} a_{11}x &+ a_{12}y = b_1 \\ a_{21}x &+ a_{22}y = b_2 \end{aligned} \qquad (1)$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are called *coefficients*, and b_1, b_2 *absolute members* of the system (1). The $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are real numbers.

Let us suppose, that the system has a solution. Applying "the addition method" we can transform the system to the form:

$$(a_{11}a_{22} - a_{12}a_{21})\,x = b_1a_{22} - b_2a_{12}$$
$$(a_{11}a_{22} - a_{12}a_{21})y = a_{11}b_2 - a_{21}b_1$$

If $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$, then the system has the only solution:

$$x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

The matrix $A = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}$ is called *matrix of the system* and the number $D = a_{11}a_{22} - a_{12}a_{21}$ is called *determinant of A* and denoted

$$D = |A| = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

If we denote $D_1 = \begin{vmatrix} b_1, & a_{12} \\ b_2, & a_{22} \end{vmatrix} = b_1a_{22} - a_{12}b_2, \quad D_2 = \begin{vmatrix} a_{11}, & b_1 \\ a_{21}, & b_2 \end{vmatrix} = a_{11}b_2 - b_1a_{21}$

then, if $D \neq 0$, we have $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$.

Analogously we can solve a system (2) of 3 linear equations in 3 unknowns:

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array} \qquad (2)$$

where $a_{ij}, b_i \in R$.

In the solution again appears a number, created by elements of the matrix

$$A = \begin{pmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{pmatrix},$$

called determinant of A and denoted by $|A|$.

Determinants of matrices of dimensions 2 (2×2) and 3 (3×3):

1. Let A be a square matrix of dimension 2. Then its determinant is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

2. Let A be a square matrix of dimension 3. Then its determinant is defined by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|,$$

where $|A_{1j}|,\ j = 1,2,3$ is determinant of a matrix obtained by deleting 1st row an jst column from A.

Remark 1. Similarly are defined determinants for matrices with dimensions > 3.
Remark 2. For evaluating determinants of matrices of the size 2×2 or 3×3, can be used a method, called "Sarus' Rule".

Solving linear Systems in 2 or 3 Unknowns
System (1) and (2) can be written in a matrix form:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The matrix $A = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}$ is called the coefficient matrix of the system (1), analogously for (2). If $b_i = 0, \quad \forall i$, then the system (1) (or (2)) is called homogeneous.

Solution of the system (1) or (2), is any ordered couple (r_1, r_2) or triple (r_1, r_2, r_3) of real numbers, resp., satisfying the system.
Two systems of equations are called equivalent one another, if they have the same set of solutions.
In solving systems of equations, we perform 4 basic operations on the system: (elementary transformations):

1. Interchanging any two equations
2. Multiplying any equation by a number $C \neq 0$
3. Multiplying any equation by a number and adding the result to another equation
4. Deleting the equation which is a multiple of another equation.

A system, obtained from the original system by applying a finite number of elementary transformation is equivalent to the primary system.

Systems (1) and (2) can be interpreted geometrically and solved graphically. From this geometric interpretation it follows: A system (1) or (2) can possess

1. the only solution

2. infinitely many solutions
3. no solution

In the case of system (1), if $D \neq 0$, there exists the only solution $(r_1, r_2) = \left(\frac{D_1}{D}, \frac{D_2}{D}\right)$. If one equation of the system (1) is a multiple of the other, system (1) has infinitely many solutions. Otherwise, there is no solution.

Linear system (2), in 3 unknowns:

1. has just one solution $(r_1, r_2, r_3) = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D}\right)$, if $D \neq 0$, where D_j, $j = 1, 2, 3$ is determinant of the matrix, obtained from the matrix A, if we replace its j-th column by absolute members.

2. has infinitely many solutions, if applying elementary transformations we obtain an equivalent system containing an equation, which is multiple of another one

3. has no solution, if its equivalent system contains an equation of the form $0 = C$, $C \neq 0$.

Remark. Formulas for the only solution of systems (1) and (2) are called Cramer's Rule.

If $D = 0$, Cramer's Rule is inapplicable and systems are solved by Gauss' Elimination Method. (GEM) GEM is a general method for solving systems of any number of equations in any number of unknowns. It consists in reduction of given system to the "triangular form", by means of a finite number of elementary transformations.

If after the reduction, the triangular system has the same number of nonzero rows as the number of unknowns, system has the only solution.
If during the reduction we obtain an equation $0 = C$, $C \neq 0$, the system has no solution.
And finally, if obtained triangular system has number of nonzero rows (equations) less then the number of unknowns and doesn't contain any equation $0 = C$, $C \neq 0$, system has infinitely many solutions.

Remark 1. Because of simplification of notation, we may perform elementary transformations on the matrix of the system (instead of on equations).

Remark 2. Since homogeneous systems never contain equations $0 = C$, $C \neq 0$ (neither systems equivalent to them), it follows that any homogeneous system, (1) or (2), always has at least trivial, zero solutions, or infinitely many solutions.