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SURFACE MEASURES REVISITED ON THE BOUNDARIES OF MATHEMATICS AND ART

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Abstract. Paper brings some reflections on attempts to confront mathematical and artistic approaches to measurements of various aspects of creativity, imagination and our understanding of aesthetic values hidden in the produced artefacts in fine art and in science, namely mathematics. Phenomena that are carriers of these subtle values enable comparison of the concepts of measures and measurement strategies in maths and art. Short revision of the surface measure definitions and its application within the two environments is presented, together with continuous efforts of finding their common features and differences.

Keywords: mathematics and art, measure, truth, beauty, inner and outer measure

Mathematics Subject Classification: Primary 28A99; Secondary 91F99, 97M80.

1 Introduction

Science and art are two highly imaginative intellectual domains of human activity with no strict boundaries dividing the spheres of competencies for one or another. In both, such outstanding abilities of the human mind as abstraction, imagination, comparison, analysis and synthesis of pieces of knowledge and perceptions, generalisation, intuition, resourcefulness, aesthetic feelings or creativity are essential and obviously used and developed. The ability to convey a message on experienced and intentionally investigated events in the form of intellectually elaborated judgments and emotionally touched up pieces of art is unique to mankind, which enables us to express, exchange, share and reclaim the outcomes of mental activities of individuals, for the advantage of the whole community. Methods of these two distinct branches of systematic intellectual activity of mankind are different so as their working approaches differ. Nevertheless, the level of abstraction, timeless validity and recency of achieved results and articulated values of the conveyed messages of these abstract disciplines provide comparably important pieces of information on perception and understanding of our world, in which we are destined to live, work and create. The products of both of these seemingly antithetical domains embody the evidence of the level of development of the human society, since they reflect the society´s latest achievements in the never-ending efforts to find and perceive the truth, search for it, and understand it as the backbone of “knowing & feeling”, undoubtedly present in harmony in both science and art.

My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful. Hermann Weyl

I am interested in mathematics only as a creative art. The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics. G.H. Hardy

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spiritual delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry. Bertrand Russell

The world around us is full of relationships, rhythms, correlations, patterns. And mathematics underlies all of these, and can be used to predict future outcomes. Our brains have evolved to survive in this world: to analyse the information it receives through our senses and spot patterns in the complexity around us. In fact, it's thought that the mathematical structure embedded in the rhythm and melody of music is what our brains latch on to, and that this is why we enjoy listening to it. It is perhaps not surprising then that there is a great deal of overlap between mathematics and the art that our brain finds so pleasing to look at.

Lewis Dartnell, [1]

Mathematical theorem is essentially an exactly formulated truth, which is a generalisation of a certain precisely recurring phenomena. It is a symbolic description of an observed behavioural feature of an object, presented in an abstract form, and simultaneously explaining also the circumstances under which the studied phenomena could appear. These circumstances are the assumptions of the proposition. Mathematical hypothesis is a symbolically coded message, often understandable only to experts, and subdued to strict laws of logics. Thus, the conveyed truth must be exactly verified by means of logical rules. It can be well comprehensible also to a layperson, when the main idea of a hypothesis is explained and illustrated, and its message can be visualised on a particular example, a geometric/graphical model preferably. Geometric interpretation helps to show the human face of mathematics. It serves as a platform for science and art matching, as a space for overlapping of the artistic and mathematical abstractions. However, their common intersection is not found only in the usage of geometric forms in fine arts. In much greater extent it can be found on the abstract level, in the search for a description of natural relations, principles and rules of our Universe.

An artist is also expressing and describing observed reality and truth by creating a masterpiece of any type – painting, musical piece, sculpture, film, or theatrical performance. Developing their own specific artistic approach she/he presents these observations in abstract form, symbolically, although "in her/his way". Reflecting subjective observations in mind and intimate emotional world, personalizing them, they are receiving a remarkable emotional mettle and generate thus the most remarkable power of the created pieces of art.

Fine art and mathematical abstract worlds are two parallels dealing with the understanding of our world and its laws, each from its own point of view and by its own means of expressions. Maths and Art represent two “on the edge” attitudes of describing the world’s truth. They seem to be the most imaginative, but in spite of this common feature, also the most respectable achievements of human mind. Nonetheless, their naturally different character in attitude and work processes is rather unifying than separating them, as these two worlds are meeting on a higher abstract level, not clearly apparent at first sight to layperson. Mathematical proposition and fine art work are two different representations of the observed reality, two abstractions appearing seemingly on the opposite poles of human mind, two distinct illustrations of truth and effort to comprise it and present in an abstract way. These two representations of "world truths" are available to complement each other perfectly.

In the following section we give some examples on how the two disciplines differ and what are the limits of their convergence. Various illustrations of pieces of art will be presented in comparison to mathematical definitions and symbolic formulas representing the concept of measurement, as the background for the assessment of understanding, truthfulness and inner value.

2 Measures

In finding a measure of any kind of unknown, we get the feeling that we have reached the first level of understanding the matter itself.

Measuring is one of the rather natural human activities, when attempting to evaluate whatever we might come across, objects we are meeting, activities we are performing and their consequences and results, our relations and their manifestations, but firstly the space in which we are compelled to live or work. Finding a measure of unknown objects or feelings, we get the impression that we understand the matter. The concept of measure is a well defined phenomenon in mathematics, and measure theory represents a complex field of mathematical theory on a highly abstract level, with verified geometric models illustrating concept of different measures as length, area, or volume of geometric figures on particular examples and calculations. What exactly is a measure in mathematics?

Let X be a non-empty set and M a non-empty system of its subsets. M is called σ-algebra, if it is closed with respect to all complements and countable unions, i.e.

$$A \in M \Rightarrow X \backslash A \in M, \;\; A_n \in M \Rightarrow \bigcup_{n=1}^{n} A_n \in M \tag{1}$$

If X is a topological space, then elements of σ-algebra B generated by system of all opened sets in X are called Borel sets.

Let M be a σ-algebra on a set X. Function $\mu : M \rightarrow [0,+\infty]$ is called (non-negative) measure, if it is σ-additive, i.e.

$$\mu(\bigcup_{n=1}^{n} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \tag{2}$$

for any arbitrary system of mutually disjoint sets, while $\{A_n\}_{n=1}^{\infty} \subset M$ and $\mu(\emptyset) = 0$. Elements of M are called measurable sets, while triple (X, M, μ) is called space with a measure.

Let $A \subset \boldsymbol{E}_n$, and let $S = \{I_1, ..., I_m\}$ be a system of closed intervals, in which no two different intervals have common interior points. Then we define $|A| = 0$, if set A does not contain interval, and if it contains some interval, then we define

$$|A|_* = \sup \sum_{k=1}^{m} |I_k|, \quad \bigcup_{k=1}^{m} I_k \subset A, \quad |A|^* = \inf \sum_{k=1}^{m} |I_k|, \quad A \subset \bigcup_{k=1}^{m} I_k \tag{3}$$

For any bounded set A the following relation is true: $|A|_* \le |A|^*$.

Number $|A|_*$ is the inner Jordan measure of the set A, number $|A|^*$ is the outer Jordan measure of the set A. Bounded set A is measurable in the Jordan sense, if $|A|_* = |A|^* = |A|$.

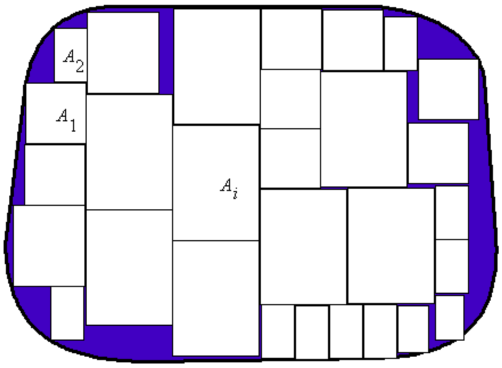

Fig. 1. Inner Jordan measure of set A - $|A|_*$.

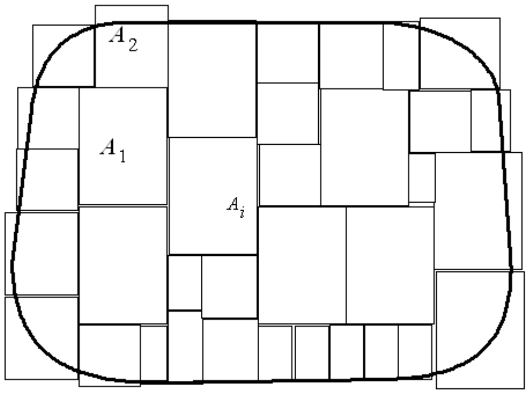

Fig. 2. Outer Jordan measure of set A - $|A|^*$.

Number $|A| \neq 0$ is called the Jordan measure of a set, it is represented as

$$|A| = \int_A \chi_A(x)dx = \int\int_A ... \int 1.dx_1 ... dx_n, \tag{4}$$

where $\chi_A(x)$ is a characteristic function such, that

$$\chi_A(x) = 1 \text{ for } x \in A, \chi_A(x) = 0 \text{ for } x \in E_n - A. \tag{5}$$

If (X, ρ) is a metric space and p is a positive number, then for any $A \subset X$ holds

$$\mu_p^*(A) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{n=1}^{\infty} \left(d(A_n)\right)^p ; A = \bigcup_{n=1}^{\infty} A_n, d(A_n) < \varepsilon, n = 1, 2, ... \right\}. \tag{6}$$

Outer measure μ_p^* is called the Hausdorff p-dimensional measure.

Measure is a certain limit that can be reached in-between two different measurements based on similar, although not completely equal principles. Limit, in which inner and outer measures meet and overlap, thus finding an agreement reflected in the real value, must not necessarily exist in all cases.

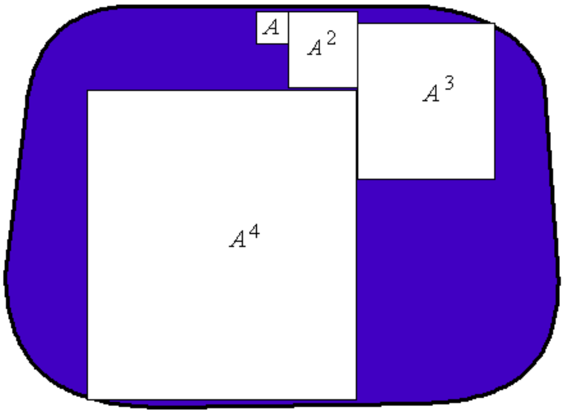

Fig. 3. Hausdorff *p*-dimensional measure of set *A*.

Fig. 4. Fractal.

Measure of an empty space is zero. The measurement units alone, objects filling the space, determine its non-zero measure. Consequently, these objects themselves, items representing our expectations, define the final measure value. The preconditions defined as assumptions influence a sort of the measure quality or type. Mathematics states these clearly by determining the initial conditions related to the specific desired type of measure, which are generally agreed and accepted, and serve as a certain guide for related measurements.

Let (X, M, μ) be a space with measure and let simultaneously X be a topological space. Measure μ is called

- Borel measure, if M contains all Borel sets, so $B \subset M$
- regular, if it is a Borel set and for all $A \in M$ holds
  - $\mu(A) = \inf\{\mu(U); A \subset U, U \text{ opened}\}$
  - $\mu(A) = \sup\{\mu(K); K \subset A, K \text{ compact}\}$
- (non-negative) Radon, if it is regular and for any compact $K \subset X$ holds $\mu(K) < \infty$
- complete, if for any $A \in M$ holds: if $\mu(A) = 0$ and $B \subset A$, then $B \in M$
- finite (or bounded), if $\mu(X) < \infty$
- σ-finite, if there exists $A_n \in M$ such, that $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$.

Several other measures can be defined, such as arithmetic, Dirac, Carathéodory or Lebesgue-Stieltjes measures, we can speak about regular, complete, finite, σ-finite, arithmetic, and many other measures, which differ in the measuring approach and the measure quality for which they are applied, see [4], [5].

Do similar laws also exist in art, or is the imagination space of an artist completely unlimited? Are there certain obstacles, defined by authors themselves, imposed maybe intuitively due to cultural development of mankind, that are influencing creative work on the masterpiece, in the sense of its pre-defined inner desirable measure? Are the different measures reflected in different artistic styles that are approaching different aims and looking for different qualities?

Painters might apply a principle similar to mathematical definition of a measure, when filling the empty two-dimensional space represented by a piece of canvas, in order to change it into a painting. According to her/his own strategy she/he starts distributing separate objects in it, thus creating a unique composition reflecting her/his own inner criteria posed onto a surface, i.e. painting measure. Author´s emotional world and creative potential influence the quality of her/his measurements, it has a strong impact on the whole composition, as it determines the inner measure of the created piece of art. Outer measure of a painting is perceived by the observers. It comes out of the 2D space of the painting and it is independent of the viewer in certain sense. Although on individual level this outer measure can be generally perceived differently, it is defined and influenced by well formed rules of our cultural heritage applicable in painting, or sculpture and design, which are developing together with mankind´s knowledge, science and culture in a historical context.

It is very interesting and inspiring to discover, how a fine artist reacts to an abstract formalised mathematical proposition from the measure theory, coded in mathematical symbolism, and how a mathematician could understand his artistic illustration of the abstract mathematical imaginations in the fine art, shaping the intuitive vision of an artist. We still know only very little about general connections between different abilities of human brain to perceive pieces of art and to understand mathematical formulas. Is the former not just an illustrative emotional visualisation and a model of the latter one? Are they interfering and cooperating, or neglecting each other passing in parallels? How can we make these two abstract abilities to respect or influence each other, and thus utilise the synergetic effect of imaginative human abilities resulting from this cooperation? Do certain general rules exist, which human beings apply when measuring space by dividing it into smaller parts that actually inhabit it, and populating it make the space alive, being the carriers of interrelation with the exterior viewer? Are all these measurement processes intuitive, innate or natural, or are they only the side effects of our material obsession and desire to evaluate all our possessions?

Some of the questioned topics can be studied in the illustrations presenting artwork of the Slovak renowned female painters of the twentieth century.

Fig. 5. Ester Šimerová Martinčeková: Country.

Fig. 6. Svetlana Ilavská: Mother.

The first presented painting in Fig. 5, "Country" by Ester Šimerová Martinčeková is an example of 2D picture with the 3D exterior measure that is visualising the 3D scene – snow-covered Slovak mountainous countryside. The chosen technique for the painting in shades of blue and grey and the simple division of the canvas area into several parts that determine the picture interior measure create in a synergic effect its 3D dimensional expression forming the outer measure of this oil painting on canvas.

The art-piece “Mother” by Svetlana Ilavská in Fig. 6 presents the unlimited power of a mother´s love. A mother, more dead than alive, carries the ill child on her back, doing her best to rescue him. The external dimension of this picture dwells in the depicted indescribable emotions - despair, belief and hope. The colour contrasts of mother and child silhouettes masterfully scratched by just few differently coloured domains in the two-dimensional space of canvas produce emotionally rich atmosphere and radiate this sensation out of the painting as its exterior dimension.

Fig. 7. Lýdia Jergušová-Vydarená: Antimass.

The third illustration in Fig. 7, “Antimass” by Lýdia Jergušová-Vydarená [4], is the artist´s visualisation of her own understanding of basic physical and mathematical concepts. The famous Einstein equation $E = mc^2$, representing physical law in mathematical symbolism, that matter can be turned into energy, and energy into matter, is presented in connection with the exploration of antimass, black matter and invisible energy concealed as the black holes of the Universe predicted by Dirac, and studied by Anderson. The artist´s imagination is expressed in a slightly adapted form of the equation with the doubled right side, thus expressing her belief in the existence of the antimass and its equivalence with the mass. Both are regarded merely as two different forms of energy that are available in the Universe. But the antimass is threatening and it makes us feel unsafe, as something unknown, mysterious, and perhaps dangerous. The sharp frame projecting from the mild background represents the antimass as the outer measure of this piece of art, which is earnestly and intrusively approaching the observer, making him/her feel insecurely. The inner measure in warm pleasant colours acts reassuringly and it calms our worry. Together they balance the story conveyed by the author in a common limit that defines the real value of the aesthetic measure of this piece of art.

The last examples in Fig. 8 represent work of young Slovak artist Žofia Dubová [2], who takes precisely into consideration the inner and the outer measure of her pieces of art. She is openly stressing the role of a visualised matter with the inner measure surrounded by the border area forming the frame, which itself is a part of the image as its outer measure. Thus, the observer is invited to perceive the artist´s message in the context of a framed object, realising both the inner and the outer measure, described explicitly by the author in the masterpiece. Experiencing both in the context of the openly articulated author´s message, gives the observer a chance to find the actual value measured as the limit balancing the inner and outer parts of the piece of art. This approach can be identified precisely with the main idea of an object measure as defined in mathematics, where both the inner and outer measures are given explicitly, e.g. in the Jordan measure definition.

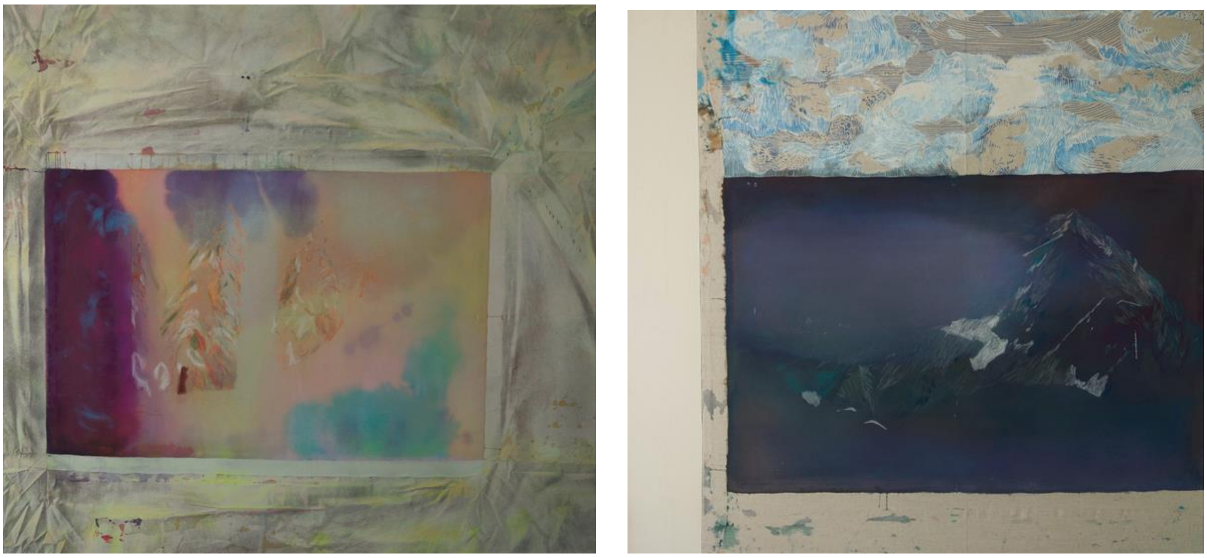

Fig. 8. Žofia Dubová: Shadows (on the left), Over the mountain (on the right).

3 Conclusions

It is not easy to balance on the edge. Boundaries are very unsafe place, where you can never know where to turn, or which direction are you falling - inside or outside? This dilemma is very openly presented in the masterpieces of Mark Rothko, Fig. 9. In his later large canvases one or more rectangular fields of colour, which bear no relationship to geometry, appear floating in the indeterminate space over the background surface, yet they never stand out fully out from it. “Green on Maroon”, masterpiece belonging to the Museo Thyssen-Bornemisza in Madrid, is a colour field painting resembling more watercolour than oil, influenced by surrealists Carl Gustav Jung, Swiss psychiatrist and psychoanalyst who founded analytical psychology. Rothko leaves door open to people to perceive different emotions. Observers have chance to choose which part of the painting has to be regarded as the interior, and which one as the exterior. In “Lavender and Mulberry”, the lavender rectangular shape is arranged in the dark rectangle so that his colour is bleeding into the dark background. Or is it the other way round? Does the maroon background intrude into the green centre? Which of the depicted rectangulars is inside the figure, and which one is extended outside? Blurred edge on the boundary of the two colours gives observers a free choice for individual perception and determination of the outer and inner measures, where colour has a specific meaning. This was the main idea of Rothko´s paintings: to evoke moods of transcendental meditation about abstraction expressions, perception of different emotions, balancing on the edge of the interior and exterior measure.

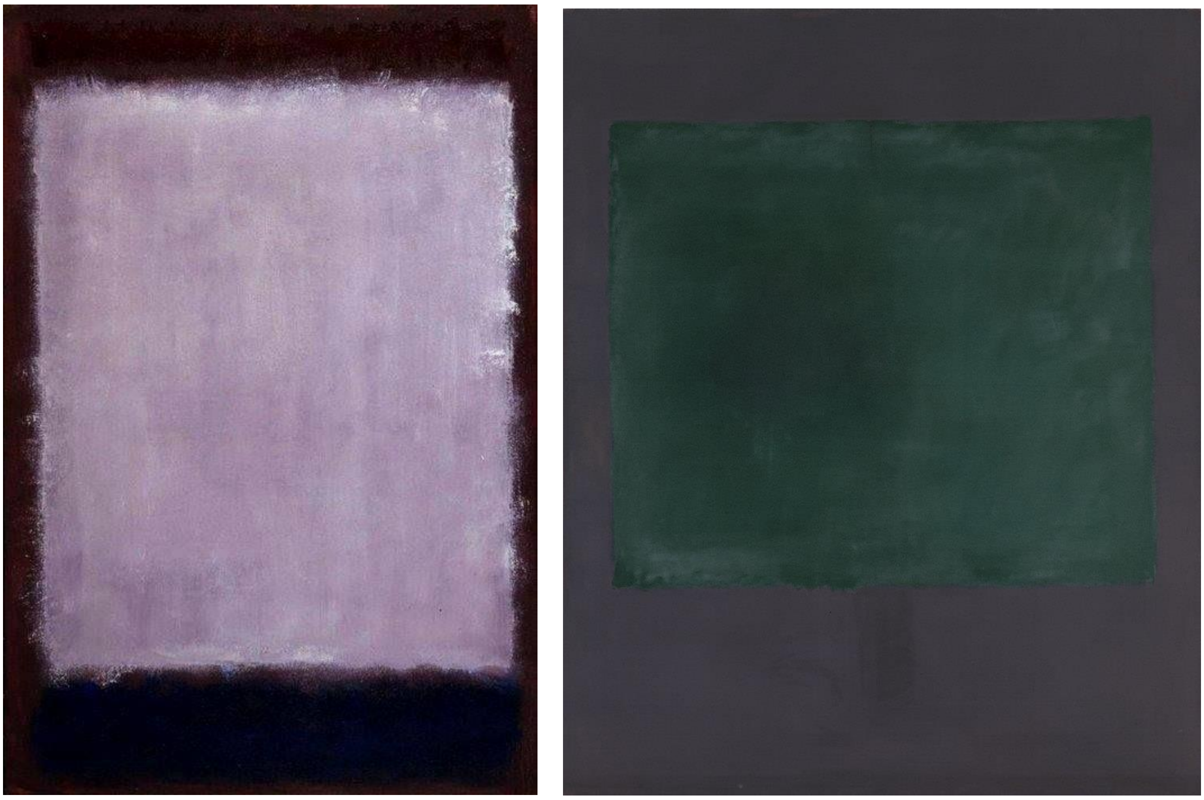

Fig. 9. Mark Rothko: Lavender and Mulberry (left), Green on Maroon (right).

It is a pursuit of elegance that captures essence, and gives us a precise insight on relations.

Xah Lee

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