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CREATIVE USE OF MATHEMATICAL STRATEGIES IN PROOFS IN UNDERGRADUATE CALCULUS

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Abstract. Mathematical theories could be developed neither without mathematical proofs, nor without mathematical creativity. Undergraduate students should be, thus, shown multiple proofs of fundamental theorems of various mathematical branches, creatively applying various techniques and strategies. In this paper we present altogether nine proofs of Lagrange's Mean Value Theorem, employing various strategies which, in general, are useful in problem solving.

Keywords: mathematical proof, mathematical creativity, Lagrange's theorem, Rolle's theorem, Cauchy's theorem

Mathematics Subject Classification: 97 I 40.

1 Introduction

More than a half-century ago the well-known mathematician and mathematics educator George Pólya (1962) "defined mathematical knowledge as information and know-how" (in: Mann, 2006, p. 237), regarding the latter one "as the more important, defining it as the ability to solve problems requiring independence, judgement, originality, and creativity" (ibid). As for the creativity in mathematical thinking, it comprises "the ability to see new relationships between techniques and areas of application, and to make associations between possibly previously unrelated ideas" (Tammadge, 1979, in: Haylock, 1987, p. 60). Furthermore, mathematical creativity manifests itself in "the independent formulation of uncomplicated mathematical problems, finding ways and means of solving these problems, the invention of proofs and theorems, the independent deduction of formulas, and finding original methods of solving nonstandard problems" (Krutetskii, 1976, in: Haylock, 1987, p. 60).
In accordance with the above presented ideas developed by various mathematics education researchers, we suggest that undergraduate calculus lectures and seminars should pay more attention to proofs of mathematical theorems in multiple ways. In addition, encouraging the students to devise their own proofs using various techniques and strategies may improve not only their understanding of the theorems and theories, but also their problem solving skills and raise the level of their mathematical creativity.

2 Diversity in proofs of Lagrange's Mean Value Theorem

Differential calculus of one real variable plays an essential role in undergraduate calculus courses – its theoretical part allows for later generalization of the theory for multivariable functions, and its application part is widely used in solutions of problems outside mathematics. Many fundamental propositions in differential calculus bear the names of recognized mathematicians, such as Pierre de Fermat, Rolle, Jean Gaston Darboux, Joseph Louis Lagrange, Augustine Louis Cauchy, Guillaume Francois Antoine de l'Hospital, Brook Taylor. In this paper we focus on Lagrange's Theorem, more precisely Lagrange's Mean Value Theorem. After introducing its wording, we present several strategies of proving the validity of the theorem.

Theorem (*Lagrange*). Let a function f be continuous over a closed interval $\langle a;b\rangle$ and differentiable over an open interval $(a;b)$. Then there exists at least one number $\xi \in (a;b)$ such that

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

Note. Geometrically speaking, the theorem says that if the function f satisfies all the assumptions, then there is a point $\xi \in (a;b)$ such that the tangent to the graph of the function f at the point $\Xi[\xi;\, f(\xi)]$ is parallel to the line $\overleftrightarrow{AB}$, where $A[a;\, f(a)]$, $B[b;\, f(b)]$ (see Fig. 1).

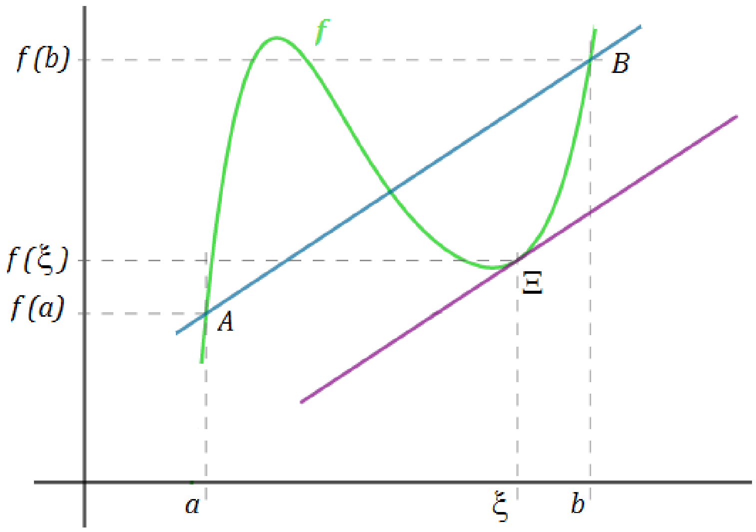

Fig. 1. Geometrical illustration of Lagrange's Mean Value Theorem..

In *Proofs 1-6*, other fundamental theorems of differential calculus and the strategy of using an auxiliary function are employed in order to prove Lagrange's Theorem. *Proof 7* shows an application of matrix determinant as a function of real variable. This proof might be of extraordinary value for university students, who, unfortunately, often misunderstand undergraduate algebra and calculus as very distant branches of mathematics. In *Proof 8* the strategy of 'special case' as an opposite of generalization is applied. *Proof 9* aims to facilitate students' understanding of Lagrange's Theorem and its obvious validity through a transition from abstract calculus to concrete interpretation in physics (or other sciences).

Proof 1.
Let us consider an auxiliary function defined as follows

$$\Phi(x) = f(x) - Kx, \text{ where } K \in \mathbf{R}.$$

The function Φ is continuous over a closed interval $\langle a; b \rangle$, and also differentiable over an open interval $(a; b)$. Taking *K* such that $\Phi(a) = \Phi(b)$ we obtain

$$K = \frac{f(b) - f(a)}{b - a}.$$

The function Φ now satisfies the assumptions of Rolle's theorem*. It follows that there must exist a $\xi \in (a; b)$ for which $\Phi'(\xi) = 0$. Since

$$\Phi'(\xi) = f'(\xi) - K = 0,$$

it is easily seen that $f'(\xi) = \frac{f(b) - f(a)}{b - a}$.

Proof 2.
A light tint of the previous proof can be obtained if the real number K is directly stated. Then, the auxiliary function can be immediately defined as

$$\Delta(x) = f(x) - \frac{f(b) - f(a)}{b - a} x,$$

and the rest of the proof is the same as shown above.

Proof 3.
Let us consider an auxiliary function defined as follows

$$\Gamma(x) = x[f(b) - f(a)] - (b - a) f(x).$$

This function is continuous over $\langle a; b \rangle$ and differentiable over $(a; b)$. It holds that

* *Rolle's Theorem:* Let a function *F* be continuous over $\langle a; b \rangle$, differentiable over $(a; b)$, and let $F(a) = F(b)$. Then there exists at least one number $\rho \in (a; b)$ such that $F'(\rho) = 0$.

$$\Gamma(a) = af(b) - bf(a) = \Gamma(b).$$

In other words, the function Γ satisfies the assumptions of Rolle's theorem, which implies that there exists a $\xi \in (a;b)$ such that $\Gamma'(\xi) = 0$. Obviously,

$$\Gamma'(\xi) = [f(b) - f(a)] - (b-a) f'(\xi) = 0,$$

which immediately implies that $f'(\xi) = \dfrac{f(b)-f(a)}{b-a}$.

Proof 4.
Indeed, the auxiliary function can assume various forms. Let us now consider

$$\Lambda(x) = f(x) - \frac{f(b)-f(a)}{b-a} x + \frac{af(b)-bf(a)}{b-a}.$$

Again, the function Λ is continuous over $\langle a;b \rangle$ and differentiable over $(a;b)$. In addition, it holds that $\Lambda(a) = 0 = \Lambda(b)$. Deriving from Rolle's theorem, there exists a $\xi \in (a;b)$ such that $\Lambda'(\xi) = 0$. Taking $x = \xi$ we get

$$\Lambda'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a},$$

and immediately $f'(\xi) = \dfrac{f(b)-f(a)}{b-a}$.

Proof 5.
Although an auxiliary function is applied in this proof again, let us refer back to the geometrical interpretation of Lagrange's Theorem (see Fig. 1). The line $\overleftrightarrow{AB}$ is described by the equation

$$y_{AB} = f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

We will investigate now the difference $\Theta(x) = f(x) - y_{AB}$, i.e.

$$\Theta(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

The function Θ is continuous over $\langle a;b \rangle$ and differentiable over $(a;b)$. Evaluating Θ at *a* and *b* we see that $\Theta(a) = 0 = \Theta(b)$, implying that Θ satisfies all the three assumptions of

Rolle's Theorem. Hence, there is at least one $\xi \in (a;b)$ such that $\Theta'(\xi)=0$. Finally, we obtain

$$\Theta'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a} = 0,$$

and consequently $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Proof 6.
By a tiny modification of the function Θ from the previous proof we get a new way to prove Lagrange's Theorem. Suffice it to consider an auxiliary function

$$\Psi(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

It can be easily justified that the function Ψ is also continuous over $\langle a;b \rangle$ and differentiable over $(a;b)$. In addition, it holds that $\Psi(a) = f(a) = \Psi(b)$. By Rolle's Theorem, there is $\xi \in (a;b)$ such that $\Psi'(x)=0$. Hence, we obtain $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Proof 7.
Besides applying geometrical interpretations, algebraic concepts can also be wisely utilized. Let us define a determinant

$$\Omega(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

This function is also continuous over $\langle a;b \rangle$ and differentiable over $(a;b)$. Following the rules of calculations with determinants it is immediate that $\Omega(a) = 0 = \Omega(b)$. Thus, by Rolle's Theorem, there is a $\xi \in (a;b)$ such that $\Omega'(\xi)=0$. More precisely, we obtain

$$\begin{vmatrix} g(a) & g(b) \\ h(a) & h(b) \end{vmatrix} f'(\xi) - \begin{vmatrix} f(a) & f(b) \\ h(a) & h(b) \end{vmatrix} g'(\xi) + \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} h'(\xi) = 0.$$

Taking $g(x) = x$ and $h(x) = 1$, the final equation assumes the form

$$\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} f'(\xi) - \begin{vmatrix} f(a) & f(b) \\ 1 & 1 \end{vmatrix} = 0, \text{ i.e. } f'(\xi) = \frac{f(b)-f(a)}{b-a}.$$

Proof 8.
The proof can be quite easily and quickly done by applying Cauchy's Theorem**. Lagrange's Theorem is, in fact, its special case. It is sufficient to take $g(x) = x$.

Proof 9.
Last but not least, if not as a mathematical proof, then surely as an intuitive consideration of validity of Lagrange's Theorem, let us employ one of its applications in elementary physics. Let $f = f(x)$ denote a function of point mass trajectory, the point changing its position from $A\left[a;\, f(a)\right]$ to $B\left[b;\, f(b)\right]$, where the continuous variable x denotes time, while $x \in \langle a;b \rangle$. Then, there must exist an instant ξ at which the instantaneous velocity, i.e. $f'(\xi)$, is equal to the average velocity, i.e. $\dfrac{f(b) - f(a)}{b-a}$.
Let us consider the motion described by the function $f = f(x)$ over the time interval $\langle a;b \rangle$. Roughly speaking, the instantaneous velocity of the point mass in motion cannot be smaller than the average velocity *over the whole interval in question*. Similarly, the instantaneous velocity of the point mass in motion cannot be greater than the average velocity at *all* instants of the considered time interval. Therefore, the instantaneous velocity of the point mass must inevitably assume values greater as well as smaller than the average velocity. Since the change of velocity over time is a continuous phenomenon, there must be an instant within the interval when the instantaneous velocity of the point mass assumes the value which is equal to the overall average velocity, i.e. there is a $\xi \in (a;b)$ such that $f'(\xi) = \dfrac{f(b) - f(a)}{b-a}$.

3 Conclusion

Every mathematical theory uses fundamental as well as less important theorems. Traditionally undergraduate calculus lectures aim to build students' knowledge of the theories gradually, systematically, including proofs of all theorems. We suggest this tradition be enriched by discussing with students multiple ways of proving the theorems. Diversity and creativity in techniques and strategies in mathematical proofs may satisfy students' needs, assuming that different students might prefer and better comprehend different approaches.

** *Cauchy's Theorem.* Let functions *F, G* be continuous over a closed interval $\langle a;b \rangle$, differentiable over an open interval $(a;b)$, and for all $x \in (a;b)$ let $g'(x) \neq 0$. Then there exists at least one $\gamma \in (a;b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\gamma)}{g'(x)}$. Incidentally, this theorem can be easily proved if we take $h(x) = 1$ in *Proof 7*.

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