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THE ROLE OF VISUALISATION IN SOLID GEOMETRY PROBLEM SOLVING

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Abstract. Visualising in mathematics is usually connected with drawing pictures or diagrams because these help learners to find a solution to the problem. The purpose of this paper is to provide qualitative research experiment on visualisation of hidden spatial objects during solid geometry problem solving. Subsequently, we offer a way how the teacher can easily demonstrate the problem via GeoGebra software and thus support the learner in problematic steps of problem solving. The usage of an interactive geometry software provides a great opportunity to demonstrate the correct image that leads to the solution.

Keywords: visualisation, mental image, mathematics problem solving, solid geometry, GeoGebra

Mathematics Subject Classification: Primary 97G40, Secondary 97D50.

1 Introduction

Students are very often afraid of solving mathematical problems. It can put a strain on some students and they can feel afraid of failing. This situation can result in loss of interest during the math lessons. However, on the other hand many teachers agree, that children like challenges. If the task is too easy, it is not challenging, so it is not interesting for them. Visualisation techniques help to solve this dilemma providing enough various tools to support mental imagery leading to the correct solution and make the task more attractive.

There are lots of research about visualisation playing an important role in the learning of mathematics. Nowadays, we can read several ideas about creating mental images and visualisation contributed by various researchers [1, 2, 3, 4, 5, 6]. However, in the early 20th century, the behaviorist revolution in psychological research led psychologists to deny the existence of mental images. From 1980s, constructivism strongly influenced the research in mathematics education. The importance of using mental imagery and visual processing in mathematics was increasingly recognised. The debate about real existence of an imprint of mental imagery in human brain took on a new form in the early 1990s, when neuroimaging became available. Recent empirical evidence [7] strongly supports the claim that humans can represent information in multiple ways, and that such representations can be used flexibly in working memory or during mental imagery.

2 Visual mental processing

The term visualisation has been used in various ways in the research literature. In [9], Presmeg gives an overview how it was used for approximately two decades. Thus visualisation is taken to include "*processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics*". This characterisation is broad enough to include two aspects of spatial thinking, namely interpreting figural information and visual processing.
Presmeg [9] identified the next five types of imagery:

Concrete imagery - picture in the mind;
Kinaesthetic imagery – physical movement;
Dynamic imagery – the image itself is moved or transformed;
Memory images of formulae;
Pattern imagery - pure relationships stripped of concrete details.

If one visualises a problem it means understanding of the problem in terms of a visual mental representation. In accordance with this statement, the visualisation process includes visual imagery as an inseparable part of the method of solution. In [10], Kosslyn describes visualisation as a cognitive process involving visual imagery in which images are either generated, inspected, transformed or used for mathematical understanding. Thus he introduces four subsequent categories of this mental process [10]:

1. Image generation,
2. Image inspection,
3. Image transformation,
4. Image use.

In general, visualisation is a mental process which results in an identified product: visual image. These mental processes and images cannot be seen, because they are in the mind of learners, but they have to be understood through the study of the mind [2]. *"Visualisation offers a method of seeing the unseen and we are encouraged and should aspire to "see" not only what comes "within sight", but also what we are unable to see."* [3].
The findings cited above have very deep implications for teaching mathematics with understanding and reasoning. Visualisation helps teachers to make instructional decisions about how to teach, about the content and the nature of tasks. Furthermore, it aids teachers with the facilitation of lessons and with the ability to engage learners in realistic situations [2].

2.1 Learners and visualisation

Educators (teachers) teach their learners (students) the content of a subject, but the question is: "What is important and more effective for them?" They should teach them to think effectively about the subject and lead them to a deeper understanding of the base of mathematical problem, the base of mathematics. Understanding how learners think during problem solving is complex and it requires special training. Makina [2] argues that all mathematics teaching should be focused on "teaching for understanding", where the visualisation is the most important basis.
In the education of mathematics, the word problem solving has its significant place and it plays a prominent role in the present. Students often have a problem with understanding the text [1] and they can not identify the relation between the known and the unknown entities.

2.2 Purposes of visualisation

Visualisation plays an important role in problem solving. During this process, we create an image of the situation or we visualise a model which helps to represent the situation mathematically. Visualising in mathematics is regularly connected with drawing pictures or diagrams. These pictures help problem solvers find a solution to the problem. However, visualisation includes more functions, such as supporting the development of ideas, facilitating communication of results and understanding. In other words, it is not only about diagrams and pictures.

A particular emphasis is sometimes placed on using visualisation during problem solving. The problem solver identifies the key components and the relationships between them.

Piggott and Woodham [4] established these three purposes for visualising:

1. **Visualising to step into the problem** – is related to the idea of "getting started". The principal role of this step is to help problem solvers with understanding what the problem is about. The visualisation enables leaners to go deeper into the situation and it also supports their understanding.
2. **Visualising to model a situation** – is especially useful to solve physically unavailable situations. Piggott mentions for example the inside of a 3D object or considering a case involving a very large number.
3. **Visualising to plan ahead** – together with the second purpose go beyond the first purpose and it suggests that visualising occurs not only at the beginning. It occurs in the middle of problem solving, too. In this case asked questions are: "What will be the consequence if we do this?", and it is connected with "What would happen if ...?". This purpose introduces using visualising during the problem solving process to anticipate a solution of a problem.

3 Visualisation of solid geometry problems

Study of geometry supports not only the logical verification, but also the spatial orientation of learners, their spatial abilities and visual perception. However, at all levels of education, teaching of solid geometry remains almost unnoticed. It often happens that the geometric knowledge of the three-dimensional space that we observe at schools is limited to the knowledge of some basic solids (cube, sphere, polyhedron and cylinder) to learn some basic formulae for the calculation of areas and volumes [5].

Andrade and Montecino [6] analysed difficulties in understanding "the really spatial" relationships when teachers use iconic planar representations instead of real models during the lesson. This way of geometry teaching often deforms the building up a correct mental picture. From studying the history of teaching, we recognise using manipulative tangible objects. Wooden or metal structures of solids have been used for better understanding of solid geometry for a long time. Nowadays, with the rapid development of digital technologies, the integration of "3D virtual objects" into education has begun to overcome several obstacles in teaching solid geometry [5, 6]. For example, dynamic program GeoGebra supports creating anaglyphs [8] from given 3D pictures, so it allows us to see (using red-green glasses) spatial objects rotating really in space in front of a computer screen. Program GeoGebra turns out to be an excellent tool for experimentation and investigation in geometry. Work with GeoGebra in education allows us direct manipulation with virtual objects and to demonstrate different geometrical relationships immediately on the computer screen. We utilised this feature of the program also in our research experiment.

4 Qualitative research experiment on visualisation of hidden spatial objects

In our experiment we focus on visualisation of a hidden spatial object (a cube appearing in a geometric-algebraic problem) by an average learner aged 17 in the 3th grade at a grammar school. The idea of the given problem was taken from [2], but we completed it to a general geometric-algebraic problem. The goal of the experiment is to detect the learner`s strategy of visualisation during the problem solving. We aspire to follow detailed steps of mental process according to Kosslyn, mentioned above, and identify the types of imagery used from Presmeg`s classification (see above). At the very beginning of the creation of the mental image, we tried to catch, according to the learner's description, what he was thinking. The next steps of the mental image creation are depicted on the learner's paper notes. According to the notes we can determine the purposes of visualisation (see Piggott's classification above).

4.1 The problem of cube layering

We designed the following problem with tasks and questions divided into four levels: *Imagine that you have an unlimited number of small cubes (all the same size* $1\times1\times1$*) in different colours. Then, imagine you will build bigger cubes from these small cubes by adding layers (like in an onion or Russian matryoshka dolls) such that each layer has a different colour.*
Next, imagine the layers of small cubes and try to answer the following questions:

1. *Imagine a cube the size of* $3\times3\times3$ *(each layer has a different colour).*
   a) *How many layers does the cube have?*
   b) *Sketch an outline how the small cubes of the outer layer touch the faces of the small cubes of the previous layers face. How many cubes of the outer layer have a face touching the inner layer faces?*
   c) *Sketch an outline how the small cubes of the outer layer touch the small cubes of the previous layer along the edges without touching any of the faces of the inner cubes. How many cubes of the outer layer touch the inner cubes along the edges?*
   d) *Sketch an outline how the small cubes of the outer layer touch the small cubes of the previous layer at the vertices without touching any of the edges of the inner cubes. How many cubes of the outer layer touch the inner cubes at the vertices?*
   e) *How many small cubes are there in total?*
   f) *Can you explain why?*
2. *Imagine a cube that has one more layer than the previous one (each layer has a different colour).*
   a) *How many small cubes did you add to the previous cube?*
   b) *Sketch an outline how the small cubes of the outer layer touch the faces of the small cubes of the previous layers face. How many cubes of the outer layer have a face touching the inner layer faces?*
   c) *Sketch an outline how the small cubes of the outer layer touch the small cubes of the previous layer along the edges without touching any of the faces of the inner cubes. How many cubes of the outer layer touch the inner cubes along the edges?*
   d) *Sketch an outline how the small cubes of the outer layer touch the small cubes of the previous layer at the vertices without touching any of the edges of the inner cubes. How many cubes of the outer layer touch the inner cubes at the vertices?*
   e) *How many small cubes are there in total?*
3. *Imagine a cube created from the cube of* $3\times3\times3$ *by adding next 5 layers. How many small cubes does the larger cube contain?*

4. *Imagine a cube created from the cube of* 3×3×3 *by adding next n layers. How many small cubes does the larger cube contain?*

The general solution of this problem is as follows:
The cube composed of k layers contains $(2k–1)^3$ small cubes.
(Remark: Adding one layer, we obtain a geometric proof of arithmetic recursive formula for odd numbers: $(2k–1)^3 + 6(2k–1)^2 + 12(2k–1) + 8 = (2k+1)^3$.)
Since the larger cube created from the cube 3×3×3 by adding next n layers has n+2 layers, it contains $(2(n+2) – 1)^3 = (2n+3)^3$ small cubes.

4.2 The learner's solution

At the beginning of solving the problem there weren't any models provided that could help a learner solve the problem. He had to "work" only with his imagination. The first substantial problem in the process of solution occurred immediately at the beginning of the problem solving.

4.2.1 Illustration of learner's solution of the task (a) at Level (1)

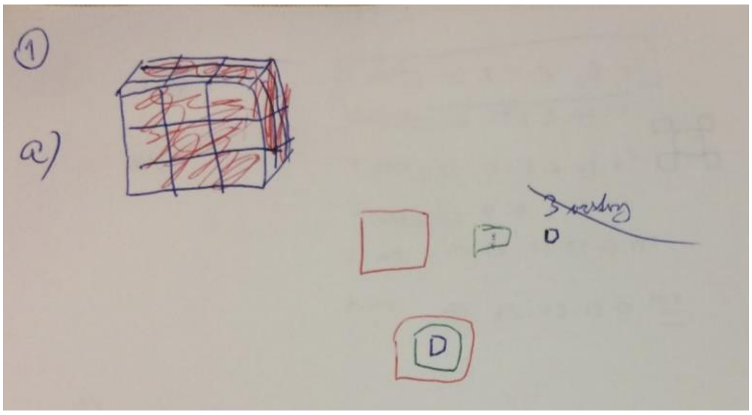


Fig. 1. Learner's solution.

The learner answered to the task (a) at Level (1) automatically, without hesitation that "The cube has 3 layers". When we asked him, if he is sure of his answer, he said, "Yes. Of course. Why not?"

(a) (b)

Fig. 2. Learner's manipulation with the Rubik's cube.

This level of the problem is the most important because it reflects the solution of the rest of the problem. After the incorrect answer, we offered him a Rubik's cube as a model of a 3×3×3 cube.

When we gave the learner the Rubik’s cube, he did not change his answer. However, the attitude of the learner changed when he started to manipulate with the floors of the Rubik’s cube (see Fig. 2). He could see the problem from a different perspective. He found that the middle floor “is rotating only around one small cube”. In fact, the middle floor of the real Rubik’s cube does not have any small cube in the centre; nevertheless, the manipulation with the Rubik’s cube helped the learner realise there could not be more layers inside.
Figure 1 represents the way the learner depicted his solution in the paper. It reflects his visual representation. The learner had problems to outline 3D objects, he prefered to sketch two-dimensional compensations, squares instead of cubes. This is the same problem as the one described by Andrade and Montecino in [6] - the learners can outline only the basic solids. When learners have to sketch a solid from different point of view, usually they do not know how to produce a correct image in their mind and how to engage their imagination into visualisation.

According to the Kosslyn’s classification this level is integrated into the mental processes Image generation and Image inspection. The learner had to create the correct mental image for correct solving of the task (a) at Level (1) as well as all the other levels. In accordance with Presmeg’s types of imagery, the learner creates Concrete imagery (picture in the mind) and applies Kinaesthetic imagery (physical movement) to solve this level. Finally, this problem is integrated into category Visualising to model a situation on the basis of the purposes of visualisation.

4.2.2 Illustration of learner’s solution of the tasks (a), (b), (c) at Level (2)

The second problem appeared in the process of solution when the learner had to change their perspective to solve the problem. In the previous case, the learner had to look inside the cube but now he had to add the next layers to the basic cube (the size of 3×3×3). In Figure 3, the learner’s solution at Level (2) is shown.

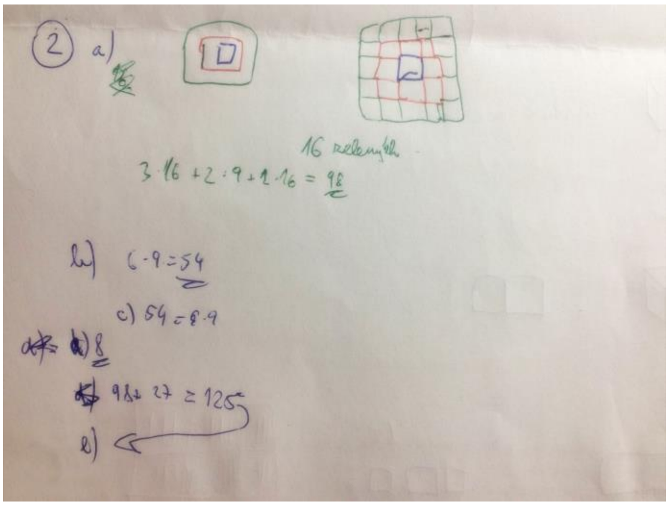


Fig. 3. Learner’s solution.

In the learner's solution the basic arithmetic operations are dominant. The learner did not outline the 3D image of the cubes, he preferred to sketch only two-dimensional images. The learner sketched his mental image only of the task (a) at Level (2) and the rest of tasks was ignored. The learner's argument for it was the following, "It is too difficult for me. I do not know how to draw connected 3D objects. We did not learn it at school."
After that the learner grabbed the Rubik's cube automatically and started to manipulate with it. He tried to describe verbally the process of adding layers to the basic cube and he said, "The results are the same as in the first level!" The learner's incorrect answer reflects that he did not change his perspective. In Level (1) he manipulated with "the outer layer" and he had to imagine the inside of the cube. In Level (2) the Rubik's cube represented "the previous layer" and the learner had to add one more layer of small cubes a different colour. The most substantial problem occurred in tasks (b) and (c) at Level (2). The learner did not become aware of the number of the added cubes. There was not only one cube per face, but, actually, there were 9 cubes on every face. The same problem occurred in the task (c) at Level (2) because there were 3 cubes on every edge.

(a)

(b)

(c)

Fig. 4. Learner's manipulation with the Rubik's cubes.

We offered the learner more Rubik's cubes and we asked him to place the cubes according to the problem levels (see Fig. 4). It is necessary to point out that this manipulation with the Rubik's cubes is not entirely correct. In fact, this placement of the cubes demostrates the addition of three layers. However, this manipulation helped the learner create better mental image and solve the problem because he considered correctly only the adjacent layer to the previous cube from the whole model.

According to the Kosslyn's classification these levels are integrated into the following mental processes: Image inspection, Image transformation and Image use. In accordance with Presmeg's types of imagery the learner creates Kinaesthetic imagery (physical movement) and Dynamic imagery (the image itself is moved or transformed) to solve these levels of the problem.

4.2.3 Illustration of learner's solution at Level (4)

The learner's solution of the task at Level (4) is not correct (see Fig. 5). The formula represents the number of small cubes of the cube the size of (n+2)×(n+2)×(n+2). However, n has to represent the number of the added layers, not the number of the small cubes.

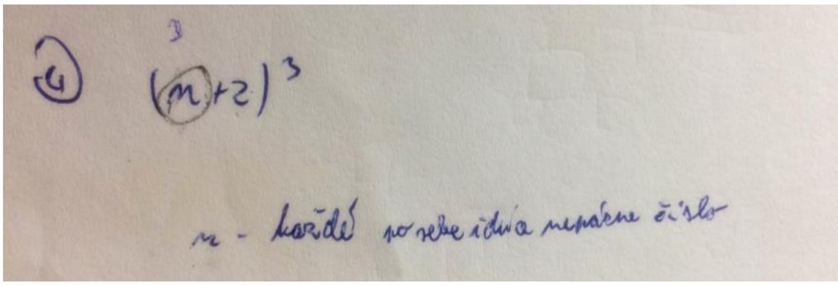


Fig. 5. Learner's solution.

In accordance with Presmeg's types of imagery, the learner creates Memory images of formulae and Pattern imagery (pure relationships stripped of concrete details) to solve these levels.

5 Using GeoGebra as a tool to create the correct mental image

Classic 3D models such as a cube, a sphere, etc. do not allow us to look at the centre of them. We can manipulate them, but we cannot break them into smaller parts. This is one of the main reasons why learners have incorrect mental images. When learners do not have the opportunity to "look to the base" of the objects, they will not know how to create a right mental image. This situation is difficult for learners, because they find themselves in a position where they are expected to imagine something that is "unimaginable" for them.

Modern information and communication technologies are powerful in representing information in different ways. This can be through text, pictures, tables and graphs or by helping visualisation of complex processes in sciences. Dynamic geometry programs allow us to manipulate objects, to change the colour or size, to rotate and to see the object from different viewpoints, etc. To solve the above-mentioned problem there is a great opportunity for using interactive geometry software. Subsequently, we offer a way how the teacher can easily demostrate the problem via GeoGebra software at Level (1) and Level (2). Figure 6 shows a possible demonstration of Level (1) via GeoGebra software.

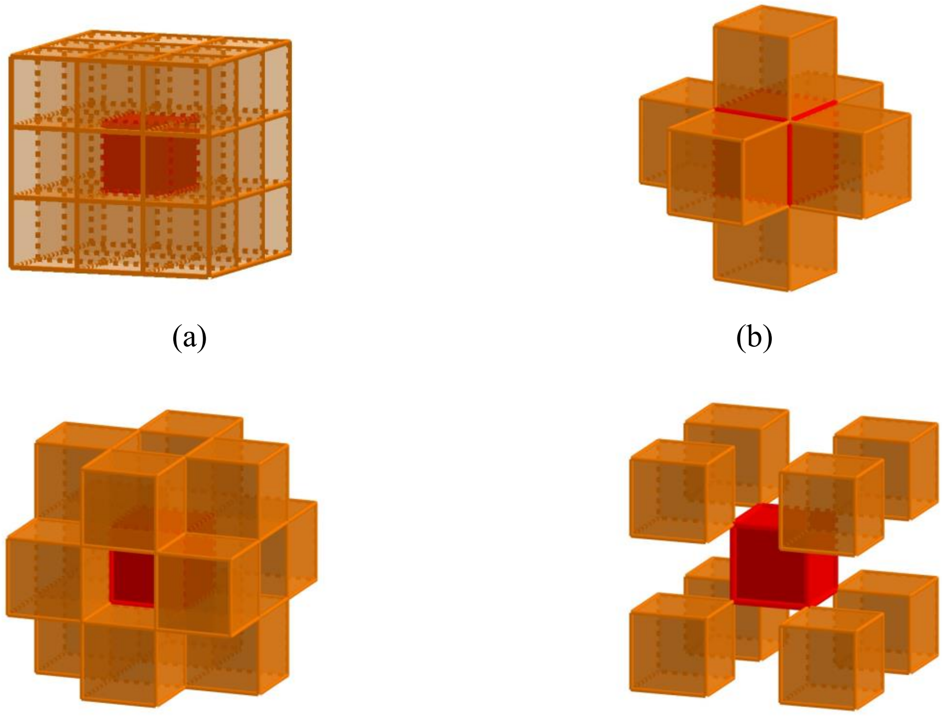
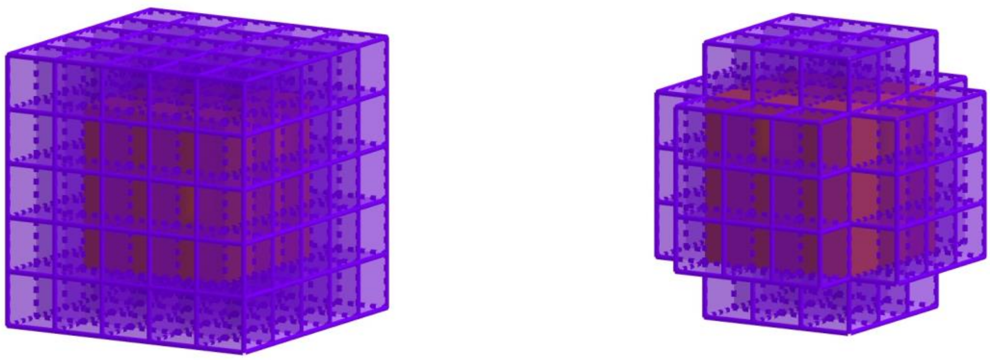
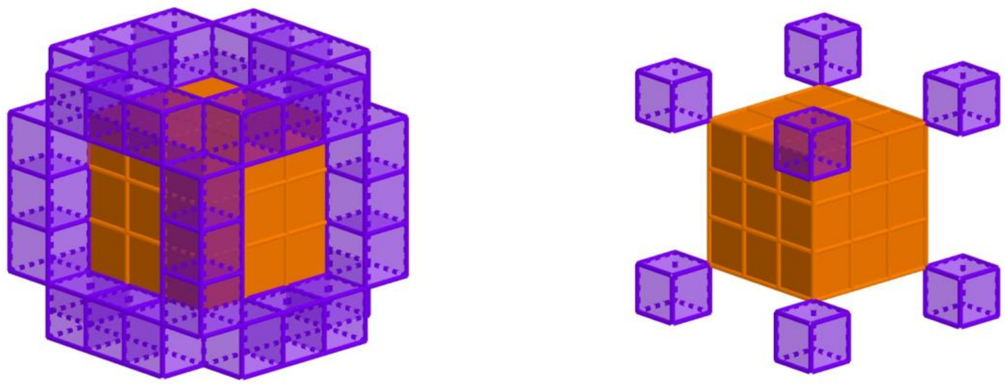


Fig. 6. The solution of the tasks at Level (1) via GeoGebra software.

Unlike Level (1), where the learner had to look inside the cube, the aim of Level (2) was to add a new layer of a different colour. The learner had to look at the cube from the outside. In other words, he had to change his perspective. Creation of new objects and manipulation with them in GeoGebra software is very simple for every learner. Thus the learners can change the perspective and they can see the object from every point of view. In this way the learners can "absorb" the correct image of the situation and create the correct mental visual image in their mind. The solution of the tasks at Level (2) via GeoGebra software (see Fig. 7 (a), (b), (c), (d)) was created by adding a new layer of purple colour small cubes to the cube in Level (1).



(a) (b)



(c) (d)

Fig. 7. The solution of the tasks at Level (2) via GeoGebra software.

Benefits of using GeoGebra software in this problem solving are the following:

- precise and ordered construction of 3D objects,
- transparency of surfaces (possibility to see all layers of the cube at the same time),
- colour differentiation of every layer,
- manipulation with cubes (possibility to change the perspective),
- demonstration of a correct image of the situation.

6 Conclusion

Qualitative research (especially case study [11]) is highly important in pedagogical sciences to see how individual students are thinking. In our experiment, we designed a solid geometry problem with hidden objects. We followed a learner`s solution step by step, focusing mainly on the mental visualisation process in this case study. We identified different categories of creating mental visual images at the learner according to appropriate theoretical literature. This helps us in preparing useful tools via GeoGebra software to support correct mental imagery in solid geometry problem solving.

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References

[1] KASHEFI, H. et al., Visualisation in mathematics problem solving meta-analysis research. In: *International Conference on Social Sciences Research,* Malaysia, v.1, 2015, e-ISBN 978-967-0792-04-0.
[2] MAKINA, A. The role of visualisation in developing critical thinking in mathematics. In: *Perspectives in Education,* South Africa, v. 28, n. 1, 2010, p. 24-33, ISSN 0258 -2236.
[3] ARCAVI, A., The role of visual representations in the learning of mathematics. In: *Educational Studies in Mathematics*, Netherlands, v. 52, n. 3, 2003, pp. 215-241, e- ISSN 1573-0816.
[4] PIGGOTT, J., WOODHAM, L. Thinking through, and by, visualising. URL, https://nrich.maths.org/6447.

[5] DEL RÍO, L., PAULETICH, F., GeoGebra como herramienta para aprender y enseñar matemática en forma dinámica. URL, http://sedici.unlp.edu.ar/bitstream/handle/10915/59903/Documento_completo.pdf-PDFA2u.pdf?sequence=3.
[6] ANDRADE, M., MONTECINO, A. La problemática de la tridimensionalidad y su representación en el plano. In: *XIII Conferencia Interamericana de Educación Matemática;Recife,* Brasil, 2011. URL, https://www.researchgate.net/publication/283684064_La_problematica_de_la_tridimensionalidad_y_su_representacion_en_el_plano.
[7] PEARSON, J., KOSSLYN, S. L., The heterogeneity of mental representation: Ending the imagery debate. In: *Proceedings of the National Academy of Sciences of the United States of America*, USA, v. 112, n. 3, 2006, ISSN 0027-8424.
[8] KMEŤOVÁ, M., VITÉZOVÁ, Z., Anaglyfy a ich využitie vo vyučovaní geometrie, In: *G-Slovenský časopis pre geometriu a grafiku,* ročník 12, č. 23, 2015: s. 31-36, ISSN 1336-524X.
[9] PRESMEG, N., Research on visualization in learning and teaching mathematics. URL, http://www.kaputcenter.umassd.edu/.
[10] KOSSLYN, M.S., *Image and brain: The resolution of the imagery debate*. 1994, London, W.W. Norton.
[11] STARMAN, A. B., The case study as a type of qualitative research. *Journal of Contemporary Educational Studies*, v. 1, 2013, pp. 28–43.

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