Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

PROTOTYPES IN PLANIMETRICS

VÁCLAVÍKOVÁ Zuzana (CZ)

Abstract. The paper is focused on the detection of a tendency of primary and secondary schools´ students to have a prototypical thought in planimetrics and consequences for mathematical thinking in geometric subjects of university students. The summary of the results which have been reached by questionnaires given to the selected schools are described (if the way used by students to sketch the specified units is similar to what they may have seen in the textbooks, eventually to results of their teachers, or if it depends on, for example, left-right orientation, sex, age or type of studied school). In addition, activities that are being prepared in this context for the students of our university study programme will be described.

Keywords: prototype, geometry, planimetrics

Mathematics Subject Classification: Primary 97-02, Secondary 97G60

1 Motivation

Let us think about this mathematical task: sketch a 5-gon ABCDE and a circumscribed circle (circumcircle) to it. Sketch a tangent line to the circle through point A.
If we do not think deeper about the task, we will automatically draw the sketch as Fig. 1 shows.

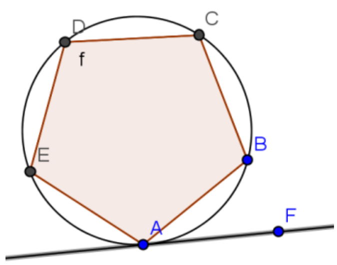

Fig. 1. 5-gon ABCDE and its circumcircle.

What is wrong? The solution is right but we have forgotten to think about the general situation. The solution should be: there is a case when it is possible to do it (the problem has a solution) but also a case when it is not possible (the problem has no solution). Because if we have a general 5-gon like Fig. 2 shows, there does not exist the circumcircle.

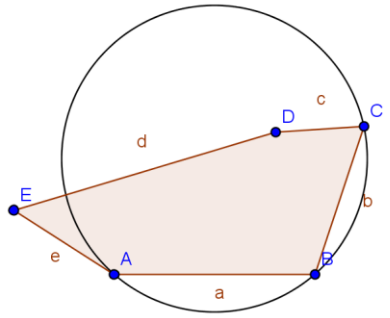

Fig. 2. General 5-gon ABCDE.

Why does it happen? The reason why we think this way is prototyping the objects. We do not use the general case for a 5-gon so often, more often we use a regular 5-gon that has a circumcircle as Fig. 1 shows. So when we hear a "5-gon", we imagine a regular one – as our "prototype" of a 5-gon.

In mathematics, like in other sciences, too strong tendency for prototyping or a bad choice of prototypes does not allow a general solution of tasks or it can significantly affect them [1], [2].

2 Prototype and theory of prototypes

A prototype, in general, is an early sample, model, or release of a product built to test a concept or process or to act as a thing to be replicated or learned from. It is an initial model of an object built to test a design. The word comes from a Greek word for "primitive form."

In context of learning, especially new terms or concepts, the prototype is the initial idea of a new, described object. If, after introducing a new concept, we begin to examine its properties and behaviour, we often follow this prototype and have a tendency to describe its properties and its behaviour.

The prototype theory is based on the notion that a given category is not defined by individual rules, but that there is some model or ideal example which characterises the given category – so-called prototype. However, a prototype may not necessarily be a real example which could be seen in real life. It can also be a mix of several other examples and such a model may appear more typical for the category than in any real creation. On the basis of similarity with the prototype, the individual decides whether or not the given object belongs to the category. Of course, the more typical the object is, the faster we can categorise it.

The prototype theory explains part of the process of knowing and classifying new concepts or objects. In our research we focused on a small area of geometry in which we tried to find out if the pupils really have prototypical ideas about some elementary geometric objects [3].

3 Research

We studied the prototyping of elementary terms in geometry between students of primary and secondary schools and potential factors of building their ideas about geometric objects - if the way used by students to sketch the specified objects is similar to what they may have seen in the textbooks, or to results of their teachers, or if it depends on, for example, left-right orientation, sex, age or type of studied school.
Testing of prototypes took place in the form of questionnaires. The basic principle in detecting prototypes is that the respondent cannot know that he/she is tested on prototypes, and in work with a given object, the prototype should be presented automatically, without deeper thinking. This is why the questionnaire was in the form of mathematical tasks where specific geometric objects were seemingly only marginal, and the text of the example concerned work or some calculation for a given object.

3.1 Preparation and creation of a questionnaire

Before the test itself, there was a discussion about the factors that could affect prototype building. The pupil's age and sex were selected, left-right orientation, assessment in mathematics, and an option for the students to inform us about a diagnosis they have (dysgraphia, dyslexia, etc.).
Together with each class, a similar questionnaire was filled in by a teacher, who also did not know that the main goal of questionnaire was testing the prototypes. It was assumed that the tendency to follow a particular prototype would be affected by the intensity of the prototype in the textbooks and by the prototype of the given object that the teacher has. The teachers also filled in the name of the textbooks which they work with in mathematical lessons.
Prior to creating the questionnaire, we also discussed whether we would require exact geometric constructions or sketches. The reason for choosing the sketches was that pupils are not influenced by laying of a ruler, so the sketches will show more about the prototype itself.
There were 8 tasks in the questionnaire from planimetrics. The tested objects were triangle, square, rectangle, right triangle, trapezoid, 5-gon, rhombus, and rhomboid.

3.2 Selected questions

As we have already mentioned, the questions were in the form of mathematical tasks so that the sketch of the prototyped object was automatic. Here are some of the questions from planimetrics:

- Sketch a triangle ABC and a circumcircle k. Sketch a tangent line to the circle through point A.
- Sketch a rhombus and write the relationship for calculating the perimeter and the area. Highlight everything you use to calculate it in your sketch.
- Sketch a right triangle, denote the sides and write the Pythagorean Theorem for it.

Every question had detailed criteria of evaluation for the prototype category of the sketch – for example the question about the triangle had four levels with respect to:

- type of the triangle (regular/equilateral triangle, isosceles triangle, acute triangle, obtuse triangle, right triangle, general, …),
- sides (lies on the side, stands at the vertice, ...),

- rotation (rotated left, right, no rotation),
- denoting the vertices (clockwise, counter-clockwise, no denoting).

Similarly, all the questions had different possible results and all questionnaires were processed manually because of the need for individual assessment of the sketches.

3.3 The results

3.3.1 Respondents

There were over 550 respondents from primary and secondary school aged 11-19 years, gender balanced, approximately 90% of students were right-oriented and 10% were left-oriented students, 2% of them wrote a diagnosis (ADHD, dysgraphia and dysortography, dyscalculia, dyslexia). The students were from primary school (144 respondents), multi-year gymnasium (104 respondents), four-year gymnasium (193 respondents) and secondary vocational school (121 respondents).

3.3.2 Selected results

Question 1: Sketch a triangle ABC and a circumcircle k. Sketch a tangent line to the circle through point A.

The question was focused on a general triangle. A triangle, as geometrical object that the pupil encounters since the youngest age, appears in the textbooks in many ways – as equilateral, isosceles, right one, convex and nonconvex, and variously rotated. We expected that pupils would have a tendency to sketch it in a general form. But the results were different. Approximately 85.71% of the respondents sketched an isosceles triangle (62.50% of the respondents sketched a regular/equilateral triangle, which is a special case of an isosceles triangle, and 23.21% of the respondents sketched an isosceles triangle that is not regular). Over 90% of the respondents sketched a triangle with the bottom side horizontally and over 86% of the respondents denoted the triangle counter-clockwise.

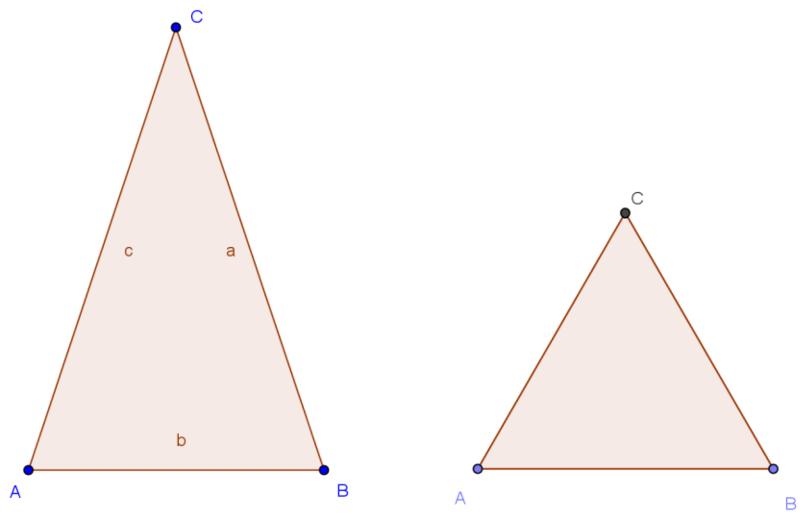

Fig. 3. Prototypes of a general triangle.

The main prototype of a general triangle is an isosceles triangle, lying on side AB parallel to the bottom edge of the paper, denoted counter-clockwise (Fig. 3 left). This case occurred in 80.54% of the questionnaires. What is more, in 57.14% of the questionnaires was sketched a regular triangle, lying on side AB parallel to the bottom edge of the paper, denoted counter-clockwise (Fig. 3 right).

Question 2: Sketch a 5-gon ABCDE and its circumcircle. Connect each two vertices to each other. How do we call the parts that have been created on the outside of the pentagon and are bounded by one side of the pentagon and a part of the circle?

A 5-gon is mentioned in the textbooks only marginally, most often in the form of a regular 5-gon. In this case, we supposed that almost all respondents would sketch a regular 5-gon, but it was sketched only in 49.84%. 18.61% of students drew the 5-gon like a "house" (see Fig. 4 right) and 13.41% drew a "diamond" (see Fig. 4 left). Three students drew a concave one. Of course, there does not exist a circumcircle, so they drew some circle through three or four points only or drew some closed curve that traversed all the vertices, but it was not a circle.

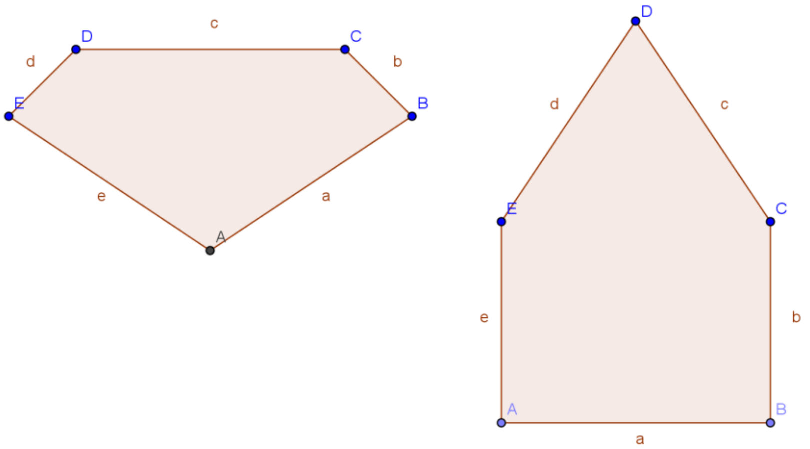

Fig. 4. Sketches of a general 5-gon.

Concerning squares and rectangles, prototypes were mostly sketched as an object with a horizontal bottom side (denoted as side AB) and counter-clockwise denoted. Much more general prototypes were in the case of not so often used objects like a trapezoid, 5-gon, rhombus, and rhomboid.

4 Conclusion of research and university students

Generally, we can state that the influence of gender, age, year, or school on the building prototypes in geometry was not confirmed.
It did not even confirm the assumption that the students with better assessment in mathematics would tend to draw geometrical objects more atypically or more generally.
There is no dependence comparing the results from the teachers' and pupils' questionnaires in the prototypes of basic objects like a triangle or square, and no dependence comparing the students' sketches with the textbooks [4].

Finally, we can proclaim that the more we know and use some kind of an object, the more generally we can think about it.
This finding should mean that university students, for whom the elementary geometric objects are well known, will have well-built prototypes and will be able to solve the tasks at a general level. But the reality is different.

5 Consequences for mathematical thinking of university students

We build up the prototypes very early, during the primary and secondary education, and that is the reason why university students have a tendency for similar prototypes in planimetrics.
At our university, we have two main study programmes – Applied Mathematics and Teacher of Mathematics. Both of them have at least three semesters of geometry and for all students of mathematics it is necessary to still keep in mind the general view of objects, because as we mentioned, the tendency for prototyping strongly affects our thinking. Especially a future teacher has to know good didactic tools how to teach this area so that his/her students will form the right prototypes. This skills and abilities need to be developed.
Textbooks for mathematical subjects for university assume that students already have a sufficient overview in the described area. But students from secondary schools are prepared to solve particular geometrical tasks that are often focused on a special case. The students are usually not prepared for discussion about generalisation of a problem with respect to other cases. Sometimes, it is simply enough to reformulate the task in such a way that will encourage discussion of various possible solutions with respect tothe object.

5.1 Mathematical tasks reflecting the prototyping

The easiest way to reformulate the tasks is using the parameters. Sometimes, it is more useful to reformulate the text in the task. Currently, we are working on finding examples of geometry tasks that support right forming of prototypes, or on converting some of the tasks to other forms to develop right prototyping. Especially in the subject Geometry Constructions, we are looking for this kind of tasks.
For example, a classical task on the construction of a circumcircle to a convex triangle can be rewritten to the following:
Draw any circle and its chord. Choose any other point on the circle and discuss the triangle made by these three points with regard to the convexity.
In this case, the student has to note that if the third point is chosen in a way that the centre of the circumcircle is outside the triangle, the triangle will be concave. If the third point is chosen in a way that the centre of the circumcircle is inside the triangle, the triangle will be convex, and if it is chosen in a way that the centre lies on the side of triangle, the triangle will be right triangle (and the circumcircle is a so-called Thales circle).
In parallel, with the students of teacher's programme in didactics subjects, we are working on collecting geometry tasks and examples for students of secondary school focused on developing the right prototypes. Future teachers discuss different examples from this point of view and learn the way how to generalise them by parameters or how to reformulate them so that the example are still understandable for secondary students but encourage discussion of various possible solutions.
At the end of our work, we hope that we will prepare a textbook with geometric examples that will reflect the prototype theory.

Acknowledgement

The paper summarizes the part of results of research that were made together with my diploma student Kristýna Šafářová and I would like to thank her for possibility to publish it.

References

[1] FERDIÁNOVÁ, V. a Poruba, J., Special 3D models for monge projection, APLIMAT 2016, 15th Conference on Applied Mathematics 2016, Proceedings, Bratislava, 2016, s. 349-361, ISBN 978-80-227-4531-4.
[2] FERDIÁNOVÁ, V., Using anaglyphs in descriptive geometry, Proceedings of the 15th European Conference on e-Learning ECEL, Praha, 2016, s. 194-200, ISBN 978-1-911218-17-3.
[3] STERNBERG, R. J., Kognitivní psychologie, Vyd. 2, Praha: Portál, 2009, ISBN 978-80-7367-638-4.
[4] ŠAFÁŘOVÁ, K., Prototypy v geometrii, diplomová práce, Ostravská univerzita, 2017.

Current address

Václavíková Zuzana, RNDr., Ph.D.
Department of Mathematics, Faculty of Science, University of Ostrava
30. dubna 22, 702 00 Ostrava, Czech Republic
E-mail: zuzana.vaclavikova@osu.cz