Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

TRAJAN'S COLUMN: MATH, ART AND CRYPTOGRAPHY

TORRE Matteo (IT)

Abstract. The paper discusses an original interpretation in cryptographic key of the continuous narrative of Trajan's Column historiated frieze, proposing a parallel between the column and the scytale, an ancient cryptographic technique.

Keywords: Roman Art, cryptography, scytale, Trajan's Column

Mathematics Subject Classification: Primary 01A20, 97A40, Secondary 97A80

1 Introduction

The history of civilization testifies to the interaction of Physics and Mathematics with Art and the influence that such disciplines have on it and on artists, for example, such as Poincaré and Penrose's theories, influenced the Escher's works on the organization of space-time, not to mention Dalí, Seurat, and many others. Feyerabend recognizes a common creative uniqueness between science and art [1] that is an effective way of science popularization. Even Ancient Rome was not insensible to the charm of art and understood its great communicative and universal character. In my opinion, one of the most significant artworks is the Trajan's Column because it is an artistic and, at the same time, scientific example of how mathematics can help art popularization to the people. Reading the art history books for high schools, it is very difficult to find the mathematical explanation of the model used for the decorative frieze (recalling the military campaigns of Trajan in Dacia), and it is even more difficult try to interpret, from a scientific point of view, the motivations that led the architect to choose that mathematical shape.

In this paper I propose a cryptographic interpretation of the mathematical model of the Trajan Column frieze. The idea born from the listening to a talk during Aplimat 2017 and the reading of the related paper [2] by Prof. Laura Tedeschini-Lalli and Prof. Alessandra Carlini. Some ideas on the mathematic interpretation of the Trajan's column frieze are taken directly from this paper [2]; while the cryptographic interpretation, supported by artistic and matematical motivations, is an original idea of mine. I am also grateful to Prof. Laura Tedeschini-Lalli and Porf. Alessandra Carlini to have been a source of inspiration, for having kindly accorded the

use of some Trajan's Column images of their copyright and for their ability to popularize with simplicity and exactness the fascinating links between science and art.

2 Ancient cryptography

The word *cryptoghrapy* comes from ancient Greek *κρυπτοσ-γραφια* which means "hidden writing", a term that denotes the art, or rather, the science, to invent methods to hide messages for military, business, commercial or just for loving purpose (lover who writes a letter to his in lover). The message, i.e. the sequence of letters we want to transmit, is called *plaintext*, while the message that is really transmitted is called *chipertext*. The process of transforming the plaintext into chipertext is called *encryption* and is done by the sender; the reverse operation is called *decryption* and is made by the receiver who knows the encryption technique: this information is the encryption *key*. Summarize in a few lines the fascinating and complex history of cryptography is not so easy, so we'll just talk about the historical aspects that are related to the themes addressed in this paper, referring those interested to the most important sources [3], [14].

The oldest example of the use of cryptography was found in some ancient Egyptian hieroglyphics carved in some monuments during the Old Kingdom and dating back more than 4,500 years ago. These are *mutili hieroglyphs*, or signs created *ex novo*, or old signs coated with acronym to make mysterious a religious text and perhaps to incite reading through decryption, thereby winning the indifference of visitors and the sense of satiety produced by the monotony of repetitive formulas. These techniques can not be considered as serious examples of secret communications, but they are to be defined as attempts at mysterious, intriguing or extravagant scriptures made by literati of that time.

The earliest informations about cryptography come from ancient Greece and are on the *spartan scytale* or *lacedemonic scytale* or, more simply, scytale (from ancient greek *σκυτάλη*, stick). The Hellenistic Greek cryptographic systems do not have any connection to the Egyptians ones: Hellenistic cryptography is essentially military (as a part of the modern cryptography), while the Egyptian crypography privileges mystical, magical and religious purposes.

Scytale is one of the oldest known transposition cipher[1], the first, and probably the only, cryptographic tool that has had success with this method. The sender choses a stick, with a certain diameter, and wrapped around a leather ribbon[2], then wrote the message from one end of the stick to the other, parallel to the axis of the stick itself. Finally, the sender unrolled the leather ribbon and sent it to the receiver who, in order to correctly read the message, had to roll the ribbon onto another stick of the same diameter[3] (Fig. 1). It is not what we today would

---

[1]A transposing cipher is an encryption method in which the positions occupied by the plaintext units are changed according to a given pattern, so that the ciphertext constitutes a permutation of the plaintext. Mathematically speaking, a two-way matching function is used on character positions during the coding operation and an inverse function during decoding.
[2]As we shall see, there are differing opinions on the material of the ribbon.
[3]The cipher key is the diameter of the stick. Another important aspect on the safety of the method is in the ribbon: if was used a leather strip, it could easily be disguised as a belt.

call a "trusty algorithm" but, according to the Greek historians, it has been used successfully for many centuries.

From Plutarch evidence in *Parallel Lives* (*Lys*, XIX, 8-12) and Aulo Gellio in *Attic Nights* (XVII, IX)[4], we know that the scytale was used by the ephors, the five supreme magistrates of Sparta, to send secret messages to the generals involved in military expeditions, confirming also that (as Tucidide recalls[5]) confidentiality was an important component of the Sparta government. Plutarch also accurately describes the cryptographic method of scytale in *Lysander's life*, where he explains that with this term we mean both the parchment with the message and the stick used for his writing and deciphering[6]. Before writing the message, the ephors prepared a long and narrow parchment strip and wound it around a stick, which was exactly the same as length and diameter to another stick that the magistrates had previously provided to the receiver. After rolling parchment to the stick, be careful not to leave any space in which the wood was visible and at the same time avoiding overlapping several edges of the parchment itself, the ephors would write the message. Then, the ephors unrolled the strip from the stick and sent it to their emissary through a messenger.

Fig. 1. Three different types of scytales: the three sticks have different diameters, this corresponds to three different cryptographic keys.

The informations from Aulus Gellius in *Attic Nights* are very similar to those of Plutarch, but from this poem we can deduce an interesting information: the strip on which the encrypted message was written was of leather. Also A. Martin [4] considers unlikely that the ribbon was made of papyrus or parchment because this materials are too breakalble; a leather band instead was more easily transportable by the messenger (who could use it as a belt) and more resistant.

The cryptographic techniques evolved rapidly, and even the ancient Romans contributed to it. Back to this time is the Caesar chiper, the first one-alphabet substitution cipher. A substitute cipher operates in this way: substitution takes place letter by letter, scrolling text from start to finish, and each letter of the plaintext is replaced in the ciphertext by the letter that there is a certain number of positions later in the alphabet (Fig. 2). The Caesar chiper takes its name from Julius Caesar, who invented it and used it during the Gaul's war to protect his secret messages.

---

[4]*Gell.*, XVII, IX, 7: *Surculi duo erant teretes, oblonguli, pari crassamento eiusdem longitudinis, derasi atque ornati consimiliter.*
[5]During the 14th year of the Peloponnesian War (March 418 - February 417), the Argives and Spartans' army clashed in the Mantinea region. Thucydides confesses his inability to count the Spartan soldiers who took part in the war because the Spartans communicated via encrypted messages (*Thuc.*, V, LXVIII, 2).
[6]More precisely, Plutarch tells about a sending of a scytale to Lisandro with the ephors immediately called him back to Sparta to justify the looting and raids he had done in the Persian satrapia of Farnabazo II, where he was at that time.
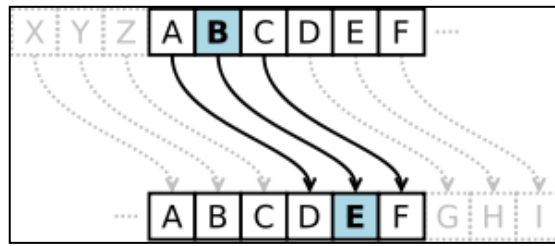
Fig. 2. mathematical diagram o f Caesar's cipher.

At that time this cryptographic technique was safe because common people were often unable to read a simple text and therefore not even a cipher one. From the discovery of alphabetic frequencies analysis, by the Arabic mathematician Al-Kindi in the XI century, all one-alphabetic substitution ciphers have become very easy to break. Over the centuries, cryptography evolved very quickly, but this brief summary of ancient cryptography is enough to the readers to understand the next paragraphs.

3 The Trajan's Column

The historical relief is the artistic type that most characterizes Roman art. It is a celebratory type used to emphasize important episodes (civil or military) of Roman history: winning campaigns, as in the Trajan and Marcus Aurelius columns, military triumphs, as in the Titus Arch, but also important events in civil life, such as the reliefs of the Trajan's Arch of Benevento, which celebrate the extension of the Appia Street to Brindisi. The celebration of military campaigns through artworks is not an exclusive of Roman art: it has precedents in Greek Hellenistic art, in which the victories of Alexander the Great are represented, through the myth and the real history, in the mosaic of Isso's battle. Typically of Roman art are, instead, some scenes such as the triumph cortege, the *adlucotio* (the speech) to the army, the sacrifice to the gods by the general, for which are fixed some canonical figurative formulas, which remain unchanged during the Roman art.

The Trajan's Column, with its long figured frieze, is among the monuments that best illustrate the strategies of political and military propaganda of the *princeps*. The Trajan's Column, built between 110 and 113 AD in the courtyard of the Ulpia Library (Fig. 3) to celebrate the two victorious campaigns of Trajan in Dacia, is a coclid column. This adjective is referred both to the spiral staircase that has internally (from Latin, *cochlis*), both for the helix resulting on their surface.
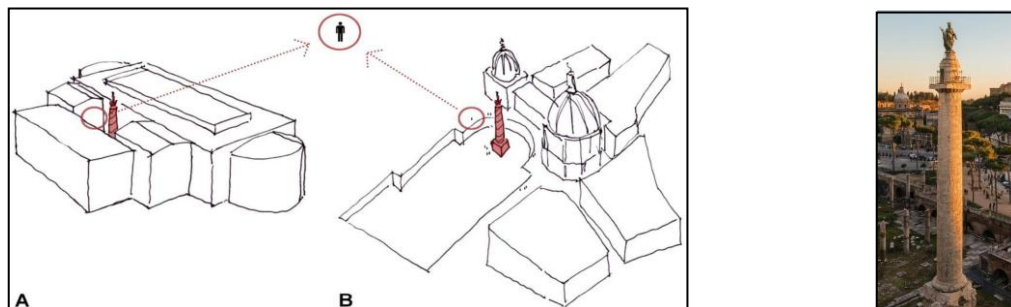

Fig. 3. on the left: onlooker at Trajan' s Column A) at the windows in the original forum complex; B) at street level in its urban context today. Image published in: A.Carlini, L.Tedeschini Lalli, Imperial roman cochlear columns and their historical narrative friezes, Proceedings Aplimat 2017, Slovak University of Technology Bratislava, Fig. N° 8 (© A. Carlini); on the right: an actual photo of Trajan's Column in Rome.

The column is a "centenary" one, that is 100 Roman feet high (29.78 meters, 39.86 meters, including the pedestal at the ground and the statue at the top) and consists of 17 colossal blocks in marble pario, each of which weighs about 40 tons and has a diameter of 3.83 meters. It has a shaft decorated with a spiral frieze, conceived as a superimposed ribbon covering it. This particular, undoubtedly mathematical nature, has created different interpretations of artistic and social nature; in the next paragraph we will also propose a cryptographic one.

In the top of the column, the frieze gives a look at the typical grooves of the Doric order. The order is, however, adapted because the column appears as a contamination between different models: a honorary column, with a statue on the top, and a votive column, adorned with reliefs in parallel bands. In addition, it associates a celebratory function of the Emperor to the funeral function, because within the base there is the cell destined to host the ashes of Trajan.

Thanks to its remarkable communication skills, the Trajan's Column has inspired the construction of other coclid columns such as the Marcus Aurelius Column, the Theodosius Column, the Arcadio's Column, the Justinian's Column, the Vendôme Column and the Alexander's Column.

The frieze: math and cryptography

The frieze of the Trajan's Column is made of bas-relief, has a length of about 200 meters and wraps 23 times around the shaft narrating the campaigns conducted by Trajan in Dacia.
The narration, realized in episodes, begins with the personification of the Danube (represented as an old man with a beard) and the passage of the Roman army on the bridge of boats (Fig. 4) and ends with the suicide of Decebalus, king of the Dacians, and the Roman conquest of the region.

In the 114 scenes, a continuous narration develops from bottom to top and places, facts and characters are described with meticulous care and the repetition of the protagonists marks the progress of the action. The rhythm with these scenes develop is so pressing that the critics agree that there is not a moment of repetitive weariness, of a repetition, of a void in the narrative context [5]. Traiano appears 59-60 times, always next to his soldiers or engaged in official acts and rituals.

Fig. 4. the passage of the Ro man army on the bridge of boats.

The frieze of the Trajan's Column can be considered as a "figurative ribbon" that wraps around the shaft of the column itself. This idea seems to be new and peculiar to Roman art.

Several hypothesis have been set forward to its genesis. We report those cited by Settis [9] and also proposed in [2]:

1. The ribbon is inspired by the *rotulo*, early form for a book, or *volumen* used in the ancient libraries and consisting of a strip of flexible material so that it could be rolled in various ways. In particular, some historians believe that the reference is to the *Commentarii*, Trajan's writings on the Dacia war, which have been lost.
2. The ribbon is inspired to the *tabulae triumphales*: pennants painted on cloths or wooden panels that were exposed during the triumphal cortege to explane to the people the places and the main episodes of the war. The ancient sources and some fragment of Roman painting make us guess that in these objects the story developed in a circular way and whit different registers.

In my opinion, the frieze of the Trajan's Column can be considered as the leather band that wraps around the wooden stick which, in this case, is represented by the column itself. For this interpretation I was ispired by the talk "*On imperial roman cochlear columns*" during the 16th Conference on Applied Mathematics [2].

The paper [2] describes in a significant and very clear way, how the narration of Trajan's military operations can be read from the bottom to the top following the continuous narration around the shaft of the column along the length of the frieze, but also following a vertical narration. In particular, the paper [2] suggests to follow three imaginary vertical line where there are aligned some significant moments: the Emperor's image, the Emperor's victory (the premonition of victory, the Winged Victory and the suicide of the Dacian king, sanctioning the defeat of the Dacians), the pontoon bridge for crossing by the troops.

For example, the very good sketch by Alessandra Carlini (Fig. 5) shows the vertical alignment of some episodes of which the lecture key is the Emperor's *adlocutio* to the army [12]. This alignment, avaiable to a standing onlooker in a particolar position, guarantees a quick interpretation of the ideological message of the frieze in accord, in my opinion, to the strategies of political and military propaganda of the *princeps*.
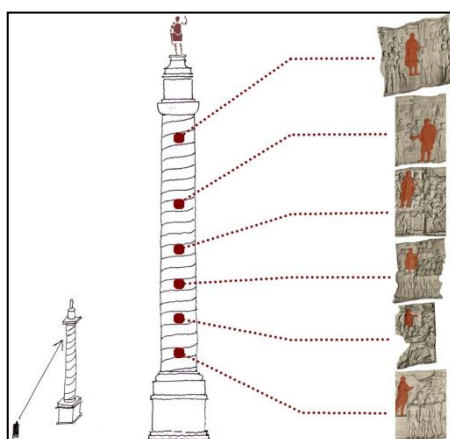

Fig. 5. Vertical alignment of Emperor's *adlocutio*, visually available to a standing onlooker.
Image published in:
A.Carlini, L.Tedeschini Lalli, *Imperial roman cochlear columns and their historical narrative friezes*, Proceedings Aplimat 2017, Slovak University of Technology Bratislava, Fig. n° 10 (© A. Carlini).

This alternative reading of the frieze narration, a-temporal and by main facts, is in my opinion closely linked to the scytale as a tool to transmit secret information for several reasons:

1. The scytale and the Trajan's Column frieze have the same particular geometry: the scytale leather ribbon and the historiated frieze are helix and their matematically construction depending on the relation between its height and the slope of the helix, as we can read in [2] and also in [6] and [11] and the bibliography thereof.

2. Cryptography was known at Roman times and adding to it the mathematical characteristics of the frieze, it is possible to support the hypothesis that the author of the column has deliberately used the diameter of the column as a reading key (a cryptographic key) to decrypt the political propaganda message of the frieze.

3. According to Salomon Reinach [13] the bas-relief is divisible into 114 parts of equal width and, therefore, there are 5 scenes at every turn. Mathematically speaking, the vertical relation among the elements s_i of a sequence $S = \{s_1, s_2, \ldots, s_{114}\}$ of 114 scenes is almost insured in the actual Trajan column (5 scenes wrapped around 23 times) by a modular arithmetic with modulus 5. So, there are 5 different vertical alignments that are visible: for $i = 1 + 5 \cdot k$, that is $i \equiv 1 (mod\ 5)$, but also for $i \equiv 0, 2, 3, 4\ (mod\ 5)$. In my opinion, not all the alignements are interesting but only those connected to the political propaganda message (whose key image can be searched in [12]), like that of the Fig. 5, and this alignments are appreciable only for one standing onlooker.

4. Replacing letters (a finite set) of the scytale leather ribbon with scenes of the Trajan's Column frieze, it is not a problem and it is possible because there are some cryptographic examples [14] where in the scytale strip the encrypted message is not a writing letter by letter but several letters (Fig. 6) are written within the breadth of the strip. These examples show again and with considerable visual impact such how the link between scytale and Trajan's Column, through helix geometry, is not so forced.
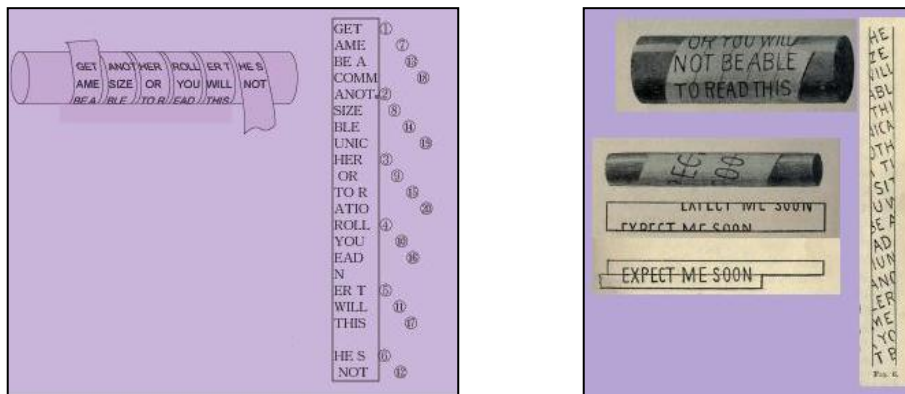

Fig. 6. Scytale as a transposition ciper by partial lines, not by letters.

Therefore becomes clear that the diameter of the column is inextricably linked to the meaning of the frieze continuous narration, but above all to the strategic position that certain scenes assume in relation to the position of the observer. Imagine to transfer the Trajan's Column frieze on the Marcus Aurelius Column (Fig. 7), which is also a centenary coclide column (that is 100 Roman feet high) but has a diameter ranging between 3.80 m and 3.65 m (which determines, therefore, a smaller number of turns of the frieze, 21 instead of 23): the frieze would lose its narrative meaning and the vertical alignment of the important scenes would no exist. It is the same situation that would happen to a spy who, intercepted the roll of skin with the encrypted message, tried to roll it on a stick with a different diameter from that used by the sender: the message would be meaningless and no interesting!

Fig. 7. Trajan's Column (on the left) and Column o f Marcus Aurelius (on the right). Image published in: A.Carlini, L.Tedeschini Lalli, *Imperial roman cochlear columns and their historical narrative friezes*, Proceedings Aplimat 2017, Slovak University of Technology Bratislava, Fig. n° 3 (© A. Carlini).

In my opinion, the motivations of the vertical alignment of the most significant scenes of the frieze can not only be linked to perspectives or artistic choices, but they must also be search in the symbolic sphere. In my opinion, for an artwork as a cochlid column, so markedly mathematical, the cryptography is an excellent link between helix geometry and art.

Conclusions and future developments

This cryptographic interpretation offers new possibilities to interpret not only the Trajan's Column, but also other cochlid columns and other monuments that contain a historiated frieze with continuous narration which I will deal with in a future paper. I want also to study in deep the analysis of the possible reading (and cryptographic) keys related to the various vertical alignments of the Trajan's Column, founding in [12], trying to verify if there are vertical alignments even for some other cochlid column.

Finally, another interesting aspect to analyze is use of scytale in other artistic monuments: I think this analysis should start with a more in-depth study of the history of the scytale.

References

[1] FEYERABEND P. THOMAS C., *Arte e Scienza*, Armando Editore, Roma, 2017.
[2] CARLINI A., TEDESCHINI-LALLI L., Imperial Roman cochlear columns and their historical narrative friezes, in *Proceedings of 16th Conference on Applied Mathematics*, APLIMAT 2017.
[3] BECKET B., *Introduction to Cryptology*, Blackwell Scientific Publications, 1988, ISBN 0-632-01836-4.
[4] MARTIN A., *Scytale*, DAGR IV, 2, 1918, p. 1162.
[5] BIANCHI BANDINELLI R., TORELLI M., *L'arte dell'antichità classica, Etruria-Roma*, Torino, Utet, 1976, ISBN 88-6008-053-3.
[6] CONVERSANO E., FRANCAVIGLIA M., LORENZI M.G., TEDESCHINI-LALLI L., Persistence of Form in Art and Architecture: Catenaries, Helicoids and Sinusoids, *APLIMAT J. of Appl. Math.*, 4 (2011).
[8] BIANCHI BANDINELLI R., *Il maestro delle imprese di Traiano*, Venezia, 2003.
[9] SETTIS S., MAFFEIS S., entry "Colonna", Enciclopedia dell'Arte Antica Treccani (1994).
[10] COLLARD B., La cryptographie dans l'Antiquité gréco-romaine. III. Le chiffrement par transposition, *Folia Electronica Classica*, Louvain-la-Neuve, N° 7, 2004.

[11] CONVERSANO E., FRANCAVIGLIA M., LORENZI M.G., TEDESCHINI-LALLI L., Geometric Forms that Persist in Art and Architecture, *Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, pp. 463-466 (http://archive.bridgesmathart.org/2011/bridges2011-463.pdf).

[12] http://www.stoa.org/trajan/indices.html

[13] REINACH S., *La Colonne Trajane au Musée de Saint-Germain: Notice et explication*, Paris, 1886 (https://mediterranees.net/civilisation/armee_romaine/colonne_trajane/reinach2.html).

[14] HULME F. E., *Cryptography or, The history, principles, and practice of cipher-writing*, London, 1898.

Current address

Torre Matteo, high school teacher, free researcher
Liceo Scientifico Statale "L. B. Alberti"
Via Pontecurone, 17, 15142 Valenza (AL), Italy
E-mail: matteo.torre1984@gmail.com