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FORCED DUFFING EQUATION WITH A NON-STRICTLY MONOTONIC POTENTIAL

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Abstract. This article is devoted to study the existence of a solution to the periodic nonlinear second order ordinary differential equation with damping

$$u''(x) + c\,u'(x) + g(x,u) = f(x)\,, \quad x \in [0,T]\,,$$
$$u(0) = u(T)\,,\ u'(0) = u'(T)\,,$$

where $c \in \mathbb{R}$, g is a Carathéodory function, $f \in L^1([0,T])$, a quotient $\frac{g(x,s)}{s}$ lies between 0 and $\frac{c^2}{4} + (\frac{\pi}{T})^2$ and a potential is a non-strictly monotonic function. The technique we use are variational method and critical point theorem.

Keywords: second order ODE, periodic problem, variational method, critical point

Mathematics subject classification: Primary 34G20; Secondary 35A15, 34K10

1 Introduction

The aim of this article is to provide new existence results for the nonlinear periodic boundary problem

$$\begin{aligned} u''(x) + c\,u'(x) + g(x,u) = f(x)\,, \quad x \in [0,T]\,,\\ u(0) = u(T)\,,\ u'(0) = u'(T)\,, \end{aligned} \tag{1.1}$$

where $c \in \mathbb{R}$, the nonlinearity $g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is Carathéodory's function, $f \in L^1([0,T])$.

An equation of this form describes for example a spring–mass system with a damper in parallel; or the charge on the capacitor in a circuit containing resistance, capacitance, and inductance.

In papers [7], [14] authors used topological degree arguments and supposed that $\gamma(x) \le \liminf_{|s|\to\infty}\left|\frac{g(x,s)}{s}\right| \le \limsup_{|s|\to\infty}\left|\frac{g(x,s)}{s}\right| \le \Gamma(x)$, $\Gamma(x) \le \left(\frac{2\pi}{T}\right)^2$ with the strict inequality on a subset of $[0,T]$ of positive measure and $\gamma(x)$ satisfies $\int_0^T \gamma(x)\,dx \ge 0$, $\int_0^T \gamma^+(x)\,dx > 0$ where $\gamma^+(x) = \max_{x\in[0,T]}\{\gamma(x), 0\}$.

But we obtain the existence result to the equation (1.1) with the nonlinearity $g(x,u) = \arctan u$ (it follows that $\gamma(x) = 0$) if the right hand side f satisfies $-\frac{\pi T}{2} < \int_0^T f(x)\,dx < \frac{\pi T}{2}$.

Others have studied problem (1.1) with jumping nonlinearities [5], [2] using also topological method.

In this article, we choose another strategy of proof which rely essentially on a variational method (see also [3], [6], [11]). We will assume that the nonlinearity g satisfies $0 \le \liminf_{|s|\to\infty}\left|\frac{g(x,s)}{s}\right| < \limsup_{|s|\to\infty}\left|\frac{g(x,s)}{s}\right| < \frac{c^2}{4} + \left(\frac{\pi}{T}\right)^2$ and a potential of g is non-strictly monotonic. Precisely, let $u = u(a,x)$, $u : \mathbb{R}\times[0,T] \to \mathbb{R}, u(a,\cdot) \in C[0,T]$ for each $a \in \mathbb{R}$, $u(\cdot,x) \in C(\mathbb{R})$ for each $x \in [0,T]$ such that $\lim_{a\to\pm\infty} u(a,x) = \pm\infty$ uniformly on $[0,T]$. We denote

$$F(s) = \int_0^T \int_0^s [\, g(x, u(a,x)) - f(x) \,]\; da\, dx\,.$$

We suppose that for such $u = u(a,x)$ there exist constants $s_1 < s_2 < s_3 < s_4$ such that

$$F(s_1) \ge F(s_2) \quad \text{and} \quad F(s_3) \le F(s_4)\,.$$

We note this assumption is fulfilled if right hand side f satisfies orthogonal condition $\int_0^T f(x)\,dx = 0$ and g satisfies sign condition $g(x,s)s \ge 0$ (Fredholm alternative for a nonlinear equation). We generalize Landesman-Lazer type condition (see [2], [12]) for a resonance problems. We get solution to (1.1) also for strong resonant problem for g satisfying $\lim_{|s|\to\infty} sg(x,s) = 0$ and $\lim_{|s|\to\infty} F(x,s) = 0$, see [8], [9].

Similarly to [1] we firstly investigate the Dirichlet problem. Then, we apply this result for finding periodic solutions. In [1] author investigate problem $u''(x) + r(x)u'(x) + g(x,u) = f(x)$ under a Lipschitz condition on g. If we rewrite this condition with $r(x) = c$ we obtain $\frac{g(x,s)-g(x,t)}{s-t} \le c_T$ where $c_T < \left(\frac{\pi}{T}\right)^2$. We will assume that $\frac{g(x,s)-g(x,t)}{s-t} \le \frac{c^2}{4} + c_T$. Hence the nonlinearity g can also cross the eigenvalue $\left(\frac{2\pi}{T}\right)^2$ if $c^2 > 12\left(\frac{\pi}{T}\right)^2$.

We note that we can use our approach also to problem with impulses, see [3] and the existence and stability of periodic solutions to problem (1.1) with $f = 0$ is discussed in [13].

2 Preliminaries

Notation: We shall use the classical space $C^k(0,T)$ of functions whose k-th derivative is continuous and the space $L^p(0,T)$ of measurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable. We denote H the Sobolev space of absolutely continuous functions $u : [0,T] \to \mathbb{R}$ such that $u' \in L^2(0,T)$, $u(0) = u(T) = 0$ endowed with the norm $\|u\| = \left(\int_0^T (u')^2\,dx\right)^{\frac{1}{2}}$.

By a solution to (1.1) we mean a function $u \in C^1(0,T)$ such that u' is absolutely continuous, u satisfies the boundary conditions and the equation (1.1) is satisfied a.e. on $(0,T)$.

Firstly we prove the existence of a solution to the Dirichlet problem

$$\begin{gathered} u''(x) + c\,u'(x) + g(x,u) = f(x)\,, \quad x \in [0,T]\,, \\ u(0) = u(T) = a\,, \end{gathered} \tag{2.1}$$

where $a \in \mathbb{R}$. Then, we apply this result for finding periodic solutions. To obtain an equation with potential we multiply (2.1) by the function $e^{\frac{c}{2}x}$. Then we put $w(x) = e^{\frac{c}{2}x}(u(x) - a)$ and get for w an equivalent problem

$$w''(x) - \frac{c^2}{4}\, w(x) + e^{\frac{c}{2}x}\, g(x, \frac{w}{e^{\frac{c}{2}x}} + a) = e^{\frac{c}{2}x}\, f(x)\,, \qquad (2.2)$$
$$w(0) = w(T) = 0\,.$$

We investigate (2.2) by using variational methods. More precisely, we find a critical point of the functional $J_a : H \to \mathbb{R}$, which is defined by

$$J_a(w) = \frac{1}{2}\int_0^T \big[(w')^2 + \frac{c^2}{4}w^2\big]\, dx - \int_0^T \big[e^{cx} G(x, \frac{w}{e^{\frac{c}{2}x}} + a) - e^{\frac{c}{2}x} f w\big]\, dx\,,$$

where

$$G(x,s) = \int_0^s g(x,t)\, dt\,.$$

We say that w_a is a critical point of J_a, if

$$\langle J_a'(w_a), z\rangle = 0 \quad \text{for all } \ z \in H.$$

We note

$$J_a(w+z) - J_a(w) = \frac{1}{2}\int_0^T \big[2w'z' + (z')^2 + \frac{c^2}{4}(2wz + z^2)\big]\, dx$$
$$- \int_0^T \big[e^{cx}\int_{\frac{w}{e^{\frac{c}{2}x}}+a}^{\frac{w+z}{e^{\frac{c}{2}x}}+a} g(x,t)\, dt - e^{\frac{c}{2}x} f z\big]\, dx\,,$$

and by mean value theorem we get

$$\int_0^T [e^{cx}\int_{\frac{w}{e^{\frac{c}{2}x}}+a}^{\frac{w+z}{e^{\frac{c}{2}x}}+a} g(x,t)\, dt]\, dx = \int_0^T [e^{cx} g(x, \xi(x))\, \frac{z}{e^{\frac{c}{2}x}}]\, dx = \int_0^T [e^{\frac{c}{2}x} g(x, \xi(x))\, z]\, dx$$

where $\xi(x) \in (\frac{w}{e^{\frac{c}{2}x}} + a, \frac{w+z}{e^{\frac{c}{2}x}} + a)$.

Therefore every critical point $w \in H$ of the functional J_a satisfies

$$\int_0^T \big[w'z' + \frac{c^2}{4}\, wz\big]\, dx - \int_0^T \big[e^{\frac{c}{2}x} g(x, \frac{w}{e^{\frac{c}{2}x}} + a) z - e^{\frac{c}{2}x} f z\big]\, dx = 0 \quad \text{for all } \ z \in H, \qquad (2.3)$$

then w is also a weak solution to the Dirichlet problem (2.2) and vice versa. The usual regularity argument for ODE proves immediately (see Fučík [4]) that any weak solution to (2.2) is also a solution in the sense mentioned above.

We remark that for any function $w \in H$ holds

$$\int_0^T \Big[(w')^2 - \Big(\frac{\pi}{T}\Big)^2 w^2\Big]\, dx \geq 0\,. \qquad (2.4)$$

We will suppose that g satisfies the following growth restrictions.

There exist functions $a_+(x), a_-(x) \in L^1(0,\pi)$ and a constant $s_0 \in \mathbb{R}^+$ such that for a.e. $x \in (0,\pi)$

$$g(x,s) \le a_-(x) \quad \text{for} \quad s \le -s_0\,, \qquad g(x,s) \ge a_+(x) \quad \text{for} \quad s \ge s_0\,, \tag{2.5}$$

(hence $0 \le \liminf\limits_{|s|\to\infty} \frac{g(x,s)}{s}$),

there exists $c_T < \left(\dfrac{\pi}{T}\right)^2$ such that

$$\frac{g(x,s)-g(x,t)}{s-t} \le \frac{c^2}{4} + c_T \qquad \text{for } s,t \in \mathbb{R}\,,\ s \ne t\,,\ x \in [0,T]\,, \tag{2.6}$$

(hence $\limsup\limits_{|s|\to\infty} \frac{g(x,s)}{s} \le \frac{c^2}{4} + c_T$ uniformly on $[0,T]$) and there are $c_1 \in \mathbb{R}^+$, $q \in L^1(0,T)$ such that

$$|g(x,s)| \le c_1|s| + q(x) \qquad \text{for all } s \in \mathbb{R}\,,\ \text{for a. e. } x \in [0,T]\,. \tag{2.7}$$

Furthermore let $u = u(a,x)$, $u : \mathbb{R} \times [0,T] \to \mathbb{R}, u(a,\cdot) \in C[0,T]$ for each $a \in \mathbb{R}$, $u(\cdot,x) \in C(\mathbb{R})$ for each $x \in [0,T]$ such that $\lim\limits_{a\to\infty} u(a,x) = +\infty$ uniformly on $[0,T]$, $\lim\limits_{a\to-\infty} u(a,x) = -\infty$ uniformly on $[0,T]$. We denote

$$F(s) = \int_0^T \int_0^s [\, g(x,u(a,x)) - f(x)\,]\; da\, dx\,.$$

We suppose that for such $u = u(a,x)$ there exist constants $s_1 < s_2 < s_3 < s_4$ such that

$$F(s_1) \ge F(s_2) \quad \text{and} \quad F(s_3) \le F(s_4)\,. \tag{2.8}$$

In this section we introduce two lemmas which will be used in the proof of the main theorem.

Lemma 2.1 *(uniqueness) Let g satisfies (2.6) then the equation (2.2) has at most one solution.*

Proof. Let $w_1, w_2 \in H$ are two solutions to (2.2) then

$$\int_0^T \left[(w_1-w_2)'z' + \frac{c^2}{4}(w_1-w_2)z\right] dx \tag{2.9}$$
$$= \int_0^T \left[e^{\frac{c}{2}x}\Big(g(x, \frac{w_1}{e^{\frac{c}{2}x}} + a) - g(x, \frac{w_2}{e^{\frac{c}{2}x}} + a)\Big)z\right] dx\,,$$

for all $z \in H$. We put $z = w_1 - w_2$ in (2.9) and using the assumption (2.6) we get

$$\int_0^T \Big((w_1-w_2)'\Big)^2 dx \le \int_0^T \Big[c_T(w_1-w_2)^2\Big]\, dx\,. \tag{2.10}$$

From (2.10), (2.4) we conclude $w_1 = w_2$.

Lemma 2.2 *(continuity) Let $a_n \to a_0$ then a corresponding sequence (w_{a_n}) solutions to the equation (2.2) with $a = a_n$ contains subsequence $(w_{a_{n_k}})$ such that $w_{a_{n_k}} \to w_0$, $w_0 \in H$ and w_0 is a solution to (2.2) with $a = a_0$.*

Proof. The solution w_{a_n} to (2.2) satisfies

$$\int_0^T \left[w'_{a_n} z' + \frac{c^2}{4}\, w_{a_n} z\right] dx - \int_0^T \left[e^{\frac{c}{2}x}\, g(x, \frac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n) z - e^{\frac{c}{2}x} f z\right] dx = 0\,, \tag{2.11}$$

for all $z \in H$.

We suppose that the sequence (w_{a_n}) is unbounded and we put $v_n = \frac{w_{a_n}}{\|w_{a_n}\|}$. Then there exists $v_0 \in H$ such that $v_n \rightharpoonup v_0$ in H and due to compact embedding H into $C([0,T])$ $v_n \to v_0$ in $C([0,T])$ (taking a subsequence if it is necessary). We divide (2.11) by $\|w_{a_n}\|$ and put $z = v_n$ then

$$\int_0^T \left[(v'_n)^2 + \frac{c^2}{4}\, v_n^2\right] dx - \int_0^T \left[\frac{e^{\frac{c}{2}x}\, g(x, \frac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n) v_n}{\|w_{a_n}\|} - \frac{e^{\frac{c}{2}x} f v_n}{\|w_{a_n}\|}\right] dx = 0\,. \tag{2.12}$$

We use inequality $\int_0^T (v'_0)^2\, dx \le \liminf\limits_{n\to\infty} \int_0^T (v'_n)^2\, dx = 1$ (the weak sequential lower semi-continuity of the Hilbert norm) and pass to the limit in (2.12). According to (2.6), (2.7) we obtain

$$\int_0^T (v'_0)^2\, dx - \int_0^T \left[c_T (v_0)^2\right] dx \le 1 - \int_0^T \left[c_T (v_0)^2\right] dx \le 0\,,$$

a contradiction to the inequality (2.4).

Therefore the sequence (w_{a_n}) is bounded. Then there exists $w_0 \in H$ such that $w_{a_n} \rightharpoonup w_0$ in H, $w_{a_n} \to w_0$ in $L^2(0,T)$, $C([0,T])$ (taking a subsequence if it is necessary).

We put $n = m$ in (2.11) and subtract this equality from (2.11) (with n) we obtain

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \left\{ \int_0^T \left[(w_{a_n} - w_{a_m})' z' + \frac{c^2}{4}\,(w_{a_n} - w_{a_m})\, z\right] dx \right. \tag{2.13}$$

$$\left. - \int_0^T \left[e^{\frac{c}{2}x} \left(g(x, \frac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n) - g(x, \frac{w_{a_m}}{e^{\frac{c}{2}x}} + a_m) \right) z\right] dx \right\} = 0\,.$$

The convergency $w_{a_n} \to w_0$ in $C([0,T])$, (2.7) and $a_n \to a_0$ yield

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^T \left[e^{\frac{c}{2}x} \Big(g(x, \frac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n) - g(x, \frac{w_{a_m}}{e^{\frac{c}{2}x}} + a_m) \Big) \left(w_{a_n} - w_{a_m}\right) \right] dx = 0\,. \tag{2.14}$$

We set $z = w_{a_n} - w_{a_m}$ in (2.13) then using (2.14) we get

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \int_0^T \left[(w'_{a_n} - w'_{a_m})^2 + \frac{c^2}{4}\,(w_{a_n} - w_{a_m})^2\right] dx = 0\,. \tag{2.15}$$

Hence the strong convergence $w_{a_n} \to w_0$ in $L^2(0,T)$ and (2.15) imply the strong convergence $w_{a_n} \to w_0$ in H and we can pass to the limit in (2.11). We obtain

$$\int_0^T \left[w'_0 z' + \frac{c^2}{4}\, w_0 z\right] dx - \int_0^T \left[e^{\frac{c}{2}x}\, g(x, \frac{w_0}{e^{\frac{c}{2}x}} + a_0) z - e^{\frac{c}{2}x} f z\right] dx = 0\,, \tag{2.16}$$

for all $z \in H$. Hence w_0 is a critical point of J_{a_0} and a solution to (2.2) with $a = a_0$.

Remark 2.1 *We have proved that to each $a \in \mathbb{R}$ there exist function $u_a = \frac{w_a}{e^{\frac{c}{2}x}} + a$ such that the $A : \mathbb{R} \to H$, $A(a) = u_a$ is a continuous operator.*

3 Main result

Theorem 3.1 *Under the assumptions (2.5), (2.6), (2.7), (2.8), Problem (1.1) has at least one solution.*

Proof. We prove that J_a is a weakly coercive functional for each $a \in \mathbb{R}$ by a contradiction. Then there is a sequence $(w_n) \subset H$ such that $\|w_n\| \to \infty$ and a constant c_2 satisfying

$$\liminf_{n\to\infty} J_a(w_n) \le c_2 \,. \tag{3.1}$$

We put $v_n = \frac{w_n}{\|w_n\|}$ then there exists $v_0 \in H$ such that $v_n \rightharpoonup v_0$ in H, $v_n \to v_0$ in $C([0,T])$. We divide (3.1) by $\|w_n\|^2$ then

$$\begin{aligned}\frac{J_a(w_n)}{\|w_n\|^2} &= \frac{1}{2}\int_0^T \Big[(v_n')^2 + \frac{c^2}{4}v_n^2\Big]\,dx - \int_0^T \Big[\frac{e^{cx}G(x, \frac{w_n}{e^{\frac{c}{2}x}} + a)}{\|w_n\|^2} - \frac{e^{\frac{c}{2}x}f}{\|w_n\|}v_n\Big]\,dx \\ &\le \frac{c_2}{\|w_n\|^2} \,.\end{aligned} \tag{3.2}$$

Due to the assumptions (2.6), (2.7) we have

$$\lim_{n\to\infty}\int_0^T \Big[\frac{e^{cx}G(x, \frac{w_n}{e^{\frac{c}{2}x}} + a)}{\|w_n\|^2}\Big]\,dx \le \frac{1}{2}\int_0^T \Big[\Big(\frac{c^2}{4} + c_T\Big)v_0^2\Big]\,dx \,. \tag{3.3}$$

Using (3.3) and passing to the limit in (3.2) we get

$$\frac{1}{2}\Big(\int_0^T \Big[(v_0')^2 - c_T\,(v_0)^2\Big]\,dx\Big) \le \frac{1}{2}\Big(1 - \int_0^T c_T(v_0)^2\,dx\Big) \le 0\,, \tag{3.4}$$

a contradiction to the inequality (2.4). Therefore J_a is a weakly coercive functional for each $a \in \mathbb{R}$.

By the standard arguments we can prove that J_a is a weakly sequentially lower semi-continuous functional on H. The weak sequential lower semi-continuity and the weak coercivity of the functional J_a imply (see Struwe [10]) the existence of a critical point w_a of the functional J_a. The usual regularity argument for ODE proves (see Fučík [4]) that w_a is also a solution to (2.2).

Now we prove the existence a classical solution to (1.1).

Let (a_n) be such sequence that $\lim\limits_{n\to\infty} a_n = \infty$ and (w_{a_n}) be a corresponding sequence of the solutions to (2.2) with $a = a_n$. We denote $w_{a_n}^+(x) = \max\{w_{a_n}(x), 0\}$, $w_{a_n}^-(x) = \max\{-w_{a_n}(x), 0\}$ and multiply equation (2.2) by $w_{a_n}^-$. Then, integrating by parts, we have

$$\int_0^T \Big[({w_{a_n}^-}')^2 + \frac{c^2}{4}(w_{a_n}^-)^2\Big]\,dx = -\int_0^T \Big[e^{\frac{c}{2}x}\,(g(x, \frac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n) - f)\,w_{a_n}^-\Big]\,dx \,. \tag{3.5}$$

By (2.6) for all $\varepsilon > 0$ there exists $r_0 > 0$ such that $g(x,s) \le (\frac{c^2}{4} + c_T + \varepsilon)s$ for all $s \ge r_0$ and $-g(x,s) \le -(\frac{c^2}{4} + c_T + \varepsilon)s$ for all $s \le -r_0$.

We can suppose that $r_0 \geq s_0$ and estimate by (2.5), (2.6), (2.7)

$$
\begin{aligned}
-\int\limits_{\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\le -r_0} \Big[e^{\frac{c}{2}x}\left(g(x,\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n)\right) w_{a_n}^-\Big]\,dx &\overset{(2.6)}{\leq} \Big(\frac{c^2}{4}+c_T+\varepsilon\Big) \int\limits_{\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\le -r_0} \Big[e^{\frac{c}{2}x}\Big(-\frac{w_{a_n}}{e^{\frac{c}{2}x}}-a_n\Big)w_{a_n}^-\Big]\,dx \\
&\leq \Big(\frac{c^2}{4}+c_T+\varepsilon\Big)\int\limits_0^T (w_{a_n}^-)^2\,dx\,,
\end{aligned}
$$

$$
\begin{aligned}
-\int\limits_{\left|\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\right|\le r_0} \Big[e^{\frac{c}{2}x}\left(g(x,\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n)\right) w_{a_n}^-\Big]\,dx &\overset{(2.7)}{\leq} \int\limits_{\left|\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\right|\le r_0} \Big[e^{\frac{c}{2}x}\Big(c_1\Big|\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\Big|+q(x)\Big)\,w_{a_n}^-\Big]\,dx \\
&\leq \int\limits_0^T [\,e^{\frac{c}{2}x}(c_1 r_0+q(x))\,w_{a_n}^-\,]\,dx\,,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
-\int\limits_{\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\ge r_0} \Big[e^{\frac{c}{2}x}\left(g(x,\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n)\right) w_{a_n}^-\Big]\,dx &\overset{(2.5)}{\leq} \int\limits_{\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n\ge r_0} \Big[e^{\frac{c}{2}x}(-a_+(x))\,w_{a_n}^-\Big]\,dx \\
&\leq \int\limits_0^T \Big[e^{\frac{c}{2}x}|a_+(x)|\,w_{a_n}^-\Big]\,dx\,.
\end{aligned}
$$

Hence there exists $Q \in L^1(0,T)$ such that

$$
\begin{aligned}
-\int_0^T&\Big[e^{\frac{c}{2}x}\left(g(x,\frac{w_{a_n}}{e^{\frac{c}{2}x}}+a_n)-f\right)w_{a_n}^-\Big]\,dx \leq \\
&\Big(\frac{c^2}{4}+c_T+\varepsilon\Big)\int_0^T (w_{a_n}^-)^2\,dx+\int_0^T Q\,w_{a_n}^-\,dx\,.
\end{aligned}
\tag{3.7}
$$

We take ε such that $\hat{c}_T := c_T+\varepsilon < \big(\frac{\pi}{T}\big)^2$. Consequently, using (3.5), (3.7) we get

$$
\Big(1-\frac{\hat{c}_T}{(\frac{\pi}{T})^2}\Big)\int_0^T\Big[({w_{a_n}^-}')^2\Big]\,dx \leq \int_0^T\Big[({w_{a_n}^-}')^2-\hat{c}_T(w_{a_n}^-)^2\Big]\,dx \leq \int_0^T\Big[Q\,w_{a_n}^-\Big]\,dx\,. \tag{3.8}
$$

This yields that for $a_n \to \infty$ the sequence $(w_{a_n}^-)$ is bounded in $C([0,T])$ (due to compact embedding H into $C([0,T])$). Similarly we obtain that the sequence $(w_{a_n}^+)$ is bounded in $C([0,T])$ for $a_n \to -\infty$.

Now we denote $u_{a_n} = \dfrac{w_{a_n}}{e^{\frac{c}{2}x}} + a_n$ then u_{a_n} is a solution to

$$
\begin{aligned}
&u''(x)+c\,u'(x)+g(x,u)=f(x)\,, \quad x\in[0,T]\,,\\
&u(0)=u(T)=a_n\,.
\end{aligned}
\tag{3.9}
$$

Since (a_n) was an arbitrary sequence and (w_a^-) is bounded in $C([0,T])$ therefore $\lim\limits_{a\to\infty} u_a(x)=\infty$ uniformly on $[0,T]$. Similarly $\lim\limits_{a\to-\infty} u_a(x)=-\infty$ uniformly on $[0,T]$.

We denote

$$F(s) = \int_0^T \int_0^s [\, g(x, u_a(x)) - f(x) \,] \; da \, dx \, .$$

Using Lemma 2.2 we conclude $F \in C(\mathbb{R}),\ F'(s) = \displaystyle\int_0^T [\, g(x, u_s) - f \,] \; dx$. We get by (2.8) (where $u(a, x) = u_a(x)$) that there exist constants $s_1 < s_2 < s_3 < s_4$ such that $F(s_1) \geq F(s_2)$ and $F(s_3) \leq F(s_4)$.

Hence there exist a constant $\hat{s} \in \mathbb{R}$ and a solution $\hat{u}$ to (3.9) with $\hat{u}(0) = \hat{u}(T) = \hat{s}$ such that $F'(\hat{s}) = \displaystyle\int_0^T [\, g(x, \hat{u}) - f \,] \; dx = 0$. Integrating (3.9) over $[0, T]$ with $u = \hat{u}$ we obtain $\displaystyle\int_0^T \hat{u}'' \, dx = 0$. Hence $\hat{u}'(0) = \hat{u}'(T)$ and the function $\hat{u}$ is a solution to the periodic problem (1.1). The proof is completed.

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References

[1] AMSTER P., Nonlinearities in a second order ODE USA-Chile Workshop on Nonlinear Analysis, Electronic Journal of Differential Equations, Conf. 06, 2001, pp. 13–21.

[2] DRÁBEK P., INVERNIZZI S., On the periodic BVP for the forced Duffing equation with jumping nonlinearity, Nonlinear Analysis 10 (1986), 643–650.

[3] DRÁBEK P., LANGEROVÁ M., On the second order periodic problem at resonance with impulses Journal of Mathematical Analysis and Applications, 428(2015) 1339—1353.

[4] FUČÍK S., Solvability of nonlinear equations and boundary value problems, D. Reidel Publishing Company, Holland 1980.

[5] HABETS P., Existence of periodic solutions of Duffing equations, Journal of Differential Equations 78 (1989), pp. 1=-32.

[6] MARIN M., On weak solutions in elasticity of dipolar bodies with voids, Journal of Computational and Applied Mathematics 82 (1997), pp. 291–297.

[7] MAWHIN J., WARD J.R., Nonuniform nonresonance condition at the two first eigenvalue for periodic solutions of forced Liénard and Duffing equations, Rocky Mountain Journal of Mathematics 12(1982), no.4, pp. 643–654.

[8] DA SILVA E. D., Quasilinear elliptic problems under strong resonance conditions, Nonlinear Analysis. 73 (2010), no. 8, 2451–2462

[9] DA SILVA E. D., Resonant elliptic problems under Cerami condition, arXiv:1205.2724.

[10] STRUWE M., Variational Methods, Springer, Berlin, (1996).

[11] TOMICZEK P., Forced Duffing equation with a resonance condition Advanced Nonlinear Studies 10 (2010), pp 573–580.

[12] TOMICZEK P., Periodic problem with a potential Landesman Lazer condition, Hindawi Publishing Corporation Boundary Value Problems (2010), Article ID 586971, 8 pages doi:10.1155/2010/586971.

[13] TORRES P.J., Existence and Stability of Periodic Solutions of a Duffing Equation by Using a New Maximum Principle, Mediterranean Journal of Mathematics, 1 (2004), 479-–486.

[14] WANG C., Multiplicity of periodic solutions for Duffing equations under nonuniform non-resonance condition, Proceedings of the American Mathematical Society, Vol. 126, No. 6, 1998, pp. 1725–1732.

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