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CYCLIC PERMUTATIONS: CROSSING NUMBERS OF THE JOIN PRODUCTS OF GRAPHS

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Abstract. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimal number of edge crossings over all drawings of G in the plane. In the paper, we extend known results concerning crossing numbers for join of graphs of order six. We give the crossing number of the join product $G + D_n$, where the graph G consists of one 4-cycle and two leaves, and D_n consists on n isolated vertices. The proof is done with the help of software that generates all cyclic permutations for a given number k, and creates a graph for a calculating the distances between all $(k-1)!$ vertices of the graph. Finally, by adding some edges to the graph G, we are able to obtain the crossing numbers of the join product with the discrete graph D_n for other graphs.

Keywords: graph, drawing, crossing number, join product, cyclic permutation

Mathematics subject classification: Primary 05C10, 05C38

1 Introduction

Let G be a simple graph with the vertex set V and the edge set E. A *drawing* of the graph G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. In such a drawing, the intersection of the interiors of the arcs is called a *crossing*. A drawing is *good* if each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a good drawing. The *crossing number* $\mathrm{cr}(G)$ of a simple graph G is defined as the minimum possible number of edge crossings in a good drawing of G in the plane. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, respectively. The join product of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$. Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $cr_D(G)$.

Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $cr_D(G_i)$. In the paper, some proofs are based on the Kleitman's result on crossing numbers of the complete bipartite graphs [12]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad \min\{m,n\} \le 6.$$

2 The crossing number of $G + D_n$

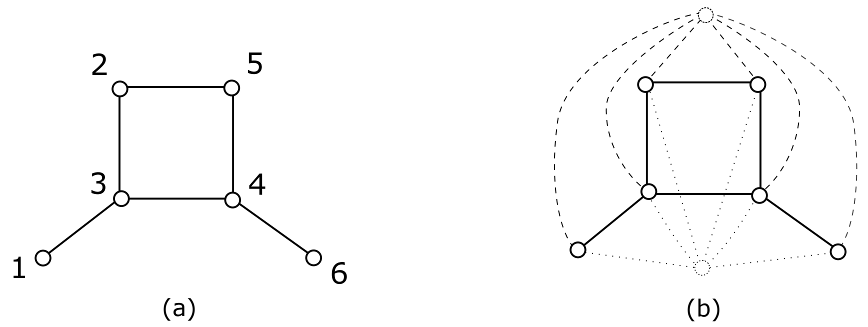


Fig. 1. Drawing of the graph G with the vertex notation and the graph $G + D_2$.

In the paper, we extent these results [4], [6], [7], [9], [10], [11] for another four graphs. Let G be the graph consisting of one 4-cycle and of two leaves. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \dots, t_n$, where any vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of G. Let T^i, $1 \le i \le n$, denote the subgraph induced by the six edges incident with the vertex t_i. Thus, $T^1 \cup \dots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$G + D_n = G \cup K_{6,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

2.1 Cyclic permutations and configurations

Let D be a good drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [3]. We emphasize that a rotation is a cyclic permutation. Hence, for $i, j \in \{1, 2, \dots, n\}$, $i \ne j$, every subgraph $T^i \cup T^j$ of the graph $G + D_n$ is isomorphic with the graph $K_{6,2}$. In the paper, we will deal with the minimum necessary number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced by the drawing D of the graph $G + D_n$ depending on the rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$.

Let D be a good drawing of the graph $K_{6,n}$. Woodall [13] proved that in the subdrawing of $T^i \cup T^j \cong K_{6,2}$ induced by D, $\mathrm{cr}_D(T^i, T^j) \ge 6$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. Moreover, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$, then $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \le \mathrm{cr}_D(T^i, T^j)$. We will separate the subgraphs T^i, $i = 1, \dots, n$, of the graph $G + D_n$ into three subsets depending on haw many the considered T^i crosses the edges of G in D. For $i = 1, 2, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \dots, n\}$. Thus, for a given drawing of

Name	Cyclic perm.	Name	Cyclic perm.	Name	Cyclic perm.	Name	Cyclic perm.
$P_1 \rightarrow$	(1 2 3 4 5 6)	$P_{31} \rightarrow$	(1 2 3 6 4 5)	$P_{61} \rightarrow$	(1 2 5 6 3 4)	$P_{91} \rightarrow$	(1 4 5 6 2 3)
$P_2 \rightarrow$	(1 3 2 4 5 6)	$P_{32} \rightarrow$	(1 3 2 6 4 5)	$P_{62} \rightarrow$	(1 5 2 6 3 4)	$P_{92} \rightarrow$	(1 5 4 6 2 3)
$P_3 \rightarrow$	(1 2 4 3 5 6)	$P_{33} \rightarrow$	(1 2 6 3 4 5)	$P_{63} \rightarrow$	(1 2 6 5 3 4)	$P_{93} \rightarrow$	(1 4 6 5 2 3)
$P_4 \rightarrow$	(1 4 2 3 5 6)	$P_{34} \rightarrow$	(1 6 2 3 4 5)	$P_{64} \rightarrow$	(1 6 2 5 3 4)	$P_{94} \rightarrow$	(1 6 4 5 2 3)
$P_5 \rightarrow$	(1 3 4 2 5 6)	$P_{35} \rightarrow$	(1 3 6 2 4 5)	$P_{65} \rightarrow$	(1 5 6 2 3 4)	$P_{95} \rightarrow$	(1 5 6 4 2 3)
$P_6 \rightarrow$	(1 4 3 2 5 6)	$P_{36} \rightarrow$	(1 6 3 2 4 5)	$P_{66} \rightarrow$	(1 6 5 2 3 4)	$P_{96} \rightarrow$	(1 6 5 4 2 3)
$P_7 \rightarrow$	(1 2 3 5 4 6)	$P_{37} \rightarrow$	(1 2 4 6 3 5)	$P_{67} \rightarrow$	(1 3 5 6 2 4)	$P_{97} \rightarrow$	(1 3 4 5 6 2)
$P_8 \rightarrow$	(1 3 2 5 4 6)	$P_{38} \rightarrow$	(1 4 2 6 3 5)	$P_{68} \rightarrow$	(1 5 3 6 2 4)	$P_{98} \rightarrow$	(1 4 3 5 6 2)
$P_9 \rightarrow$	(1 2 5 3 4 6)	$P_{39} \rightarrow$	(1 2 6 4 3 5)	$P_{69} \rightarrow$	(1 3 6 5 2 4)	$P_{99} \rightarrow$	(1 3 5 4 6 2)
$P_{10} \rightarrow$	(1 5 2 3 4 6)	$P_{40} \rightarrow$	(1 6 2 4 3 5)	$P_{70} \rightarrow$	(1 6 3 5 2 4)	$P_{100} \rightarrow$	(1 5 3 4 6 2)
$P_{11} \rightarrow$	(1 3 5 2 4 6)	$P_{41} \rightarrow$	(1 4 6 2 3 5)	$P_{71} \rightarrow$	(1 5 6 3 2 4)	$P_{101} \rightarrow$	(1 4 5 3 6 2)
$P_{12} \rightarrow$	(1 5 3 2 4 6)	$P_{42} \rightarrow$	(1 6 4 2 3 5)	$P_{72} \rightarrow$	(1 6 5 3 2 4)	$P_{102} \rightarrow$	(1 5 4 3 6 2)
$P_{13} \rightarrow$	(1 2 4 5 3 6)	$P_{43} \rightarrow$	(1 3 4 6 2 5)	$P_{73} \rightarrow$	(1 2 4 5 6 3)	$P_{103} \rightarrow$	(1 3 4 6 5 2)
$P_{14} \rightarrow$	(1 4 2 5 3 6)	$P_{44} \rightarrow$	(1 4 3 6 2 5)	$P_{74} \rightarrow$	(1 4 2 5 6 3)	$P_{104} \rightarrow$	(1 4 3 6 5 2)
$P_{15} \rightarrow$	(1 2 5 4 3 6)	$P_{45} \rightarrow$	(1 3 6 4 2 5)	$P_{75} \rightarrow$	(1 2 5 4 6 3)	$P_{105} \rightarrow$	(1 3 6 4 5 2)
$P_{16} \rightarrow$	(1 5 2 4 3 6)	$P_{46} \rightarrow$	(1 6 3 4 2 5)	$P_{76} \rightarrow$	(1 5 2 4 6 3)	$P_{106} \rightarrow$	(1 6 3 4 5 2)
$P_{17} \rightarrow$	(1 4 5 2 3 6)	$P_{47} \rightarrow$	(1 4 6 3 2 5)	$P_{77} \rightarrow$	(1 4 5 2 6 3)	$P_{107} \rightarrow$	(1 4 6 3 5 2)
$P_{18} \rightarrow$	(1 5 4 2 3 6)	$P_{48} \rightarrow$	(1 6 4 3 2 5)	$P_{78} \rightarrow$	(1 5 4 2 6 3)	$P_{108} \rightarrow$	(1 6 4 3 5 2)
$P_{19} \rightarrow$	(1 3 4 5 2 6)	$P_{49} \rightarrow$	(1 2 3 5 6 4)	$P_{79} \rightarrow$	(1 2 4 6 5 3)	$P_{109} \rightarrow$	(1 3 5 6 4 2)
$P_{20} \rightarrow$	(1 4 3 5 2 6)	$P_{50} \rightarrow$	(1 3 2 5 6 4)	$P_{80} \rightarrow$	(1 4 2 6 5 3)	$P_{110} \rightarrow$	(1 5 3 6 4 2)
$P_{21} \rightarrow$	(1 3 5 4 2 6)	$P_{51} \rightarrow$	(1 2 5 3 6 4)	$P_{81} \rightarrow$	(1 2 6 4 5 3)	$P_{111} \rightarrow$	(1 3 6 5 4 2)
$P_{22} \rightarrow$	(1 5 3 4 2 6)	$P_{52} \rightarrow$	(1 5 2 3 6 4)	$P_{82} \rightarrow$	(1 6 2 4 5 3)	$P_{112} \rightarrow$	(1 6 3 5 4 2)
$P_{23} \rightarrow$	(1 4 5 3 2 6)	$P_{53} \rightarrow$	(1 3 5 2 6 4)	$P_{83} \rightarrow$	(1 4 6 2 5 3)	$P_{113} \rightarrow$	(1 5 6 3 4 2)
$P_{24} \rightarrow$	(1 5 4 3 2 6)	$P_{54} \rightarrow$	(1 5 3 2 6 4)	$P_{84} \rightarrow$	(1 6 4 2 5 3)	$P_{114} \rightarrow$	(1 6 5 3 4 2)
$P_{25} \rightarrow$	(1 2 3 4 6 5)	$P_{55} \rightarrow$	(1 2 3 6 5 4)	$P_{85} \rightarrow$	(1 2 5 6 4 3)	$P_{115} \rightarrow$	(1 4 5 6 3 2)
$P_{26} \rightarrow$	(1 3 2 4 6 5)	$P_{56} \rightarrow$	(1 3 2 6 5 4)	$P_{86} \rightarrow$	(1 5 2 6 4 3)	$P_{116} \rightarrow$	(1 5 4 6 3 2)
$P_{27} \rightarrow$	(1 2 4 3 6 5)	$P_{57} \rightarrow$	(1 2 6 3 5 4)	$P_{87} \rightarrow$	(1 2 6 5 4 3)	$P_{117} \rightarrow$	(1 4 6 5 3 2)
$P_{28} \rightarrow$	(1 4 2 3 6 5)	$P_{58} \rightarrow$	(1 6 2 3 5 4)	$P_{88} \rightarrow$	(1 6 2 5 4 3)	$P_{118} \rightarrow$	(1 6 4 5 3 2)
$P_{29} \rightarrow$	(1 3 4 2 6 5)	$P_{59} \rightarrow$	(1 3 6 2 5 4)	$P_{89} \rightarrow$	(1 5 6 2 4 3)	$P_{119} \rightarrow$	(1 5 6 4 3 2)
$P_{30} \rightarrow$	(1 4 3 2 6 5)	$P_{60} \rightarrow$	(1 6 3 2 5 4)	$P_{90} \rightarrow$	(1 6 5 2 4 3)	$P_{120} \rightarrow$	(1 6 5 4 3 2)

Tab. 1. Names of Cyclic Permutations of 6-elements set.

G, any F^i is exactly represented by $\text{rot}_D(t_i)$. All cyclic permutations of six elements can be generated by the algorithm [2], and they are collected in Tab. 1.

We will dealt with only drawings of the graph G with a possibility of an existence of a subgraph $T_i \in R_D$ because of arguments in the proof of the main Theorem 1. Assume a good drawing D of the graph $G + D_n$ in which the edges of G does not cross each other. In this case, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Fig. 1(a). It is easy to see that, in D, there are only four different possible configurations of F^i summarized in Tab. 2. In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. As for our considerations does not play role which of the regions is unbounded, assume the drawings shown in Figure 2. In a fixed drawing of the graph $G + D_n$, some configurations from the set $\mathcal{M} = \{A_1, A_2, A_3, A_4\}$ do not must appear. We denote by $\mathcal{M}_D$ the set of all configurations that exist in the drawing D belonging to $\mathcal{M}$.

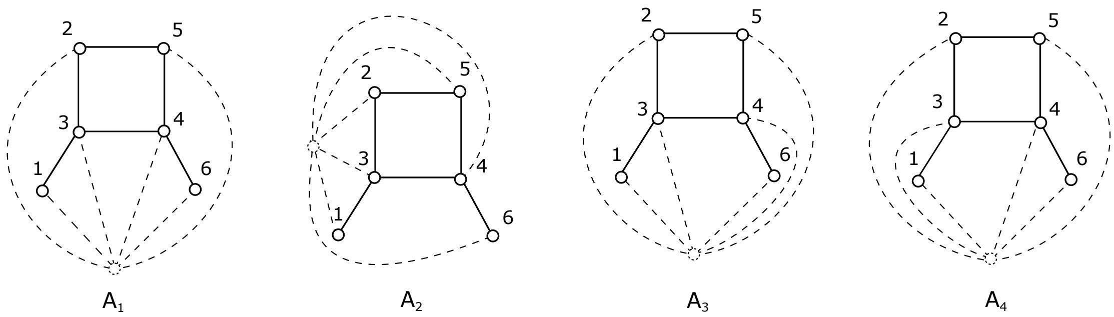


Fig. 2. Drawings of four possible configurations of graph F^i with the vertices of G denoted as in Fig. 1(a).

$A_1 : (125643)$	$A_2 : (132546)$
$A_3 : (125463)$	$A_4 : (132564)$

Tab. 2. Configurations of graph F^i with the vertices of G denoted as in Fig. 1(a).

$-$	A_1	A_2	A_3	A_4
A_1	6	4	5	5
A_2	4	6	5	5
A_3	5	5	6	5
A_4	5	5	5	6

Tab. 3. Lower-bounds of numbers of crossings for two configurations from $\mathcal{M}$.

Let X, Y be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(X,Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X,Y) = \min\{\mathrm{cr}_D(X,Y)\}$ over all good drawings of the graph $G + D_n$. In the next statements we are able to use the possibilities of the algorithm of the cyclic permutations of 6-elements set, see [2]. By $\overline{P_i}$ we will understand the inverse cyclic permutation to the permutation P_i, for $i = 1, \dots, 120$. Woodall [13] defined the cyclic-ordered graph COG with the set of vertices $V = \{P_1, P_2, \dots, P_{120}\}$, and with the set of edges E, where two vertices are joined by the edge if the vertices correspond to the permutations P_i and P_j, which are formed by the exchange of exactly two adjacent elements of the 6-tuple (i. e. an ordered set with 6 elements). Hence, if $d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\mathrm{rot}_D(t_j)\text{"})$ denotes the distance between two vertices correspond to the cyclic permutations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$ in the graph COG, then

$$d_{COG}(\text{"}\mathrm{rot}_D(t_i)\text{"}, \text{"}\overline{\mathrm{rot}_D(t_j)}\text{"}) = Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \leq \mathrm{cr}_D(T^i, T^j)$$

for any two different subgraphs T^i and T^j. The configurations A_1 and A_2 are represented by the cyclic permutations $P_{85} = (125643)$ and $P_8 = (132546)$, respectively. Using $\overline{P_8} = (164523) = P_{94}$ and $d_{COG}(\text{"}P_{85}\text{"}, \text{"}P_{94}\text{"}) = 4$ we obtain $\mathrm{cr}(A_1, A_2) \geq 4$. The same reason gives $\mathrm{cr}(A_1, A_3) \geq 5$, $\mathrm{cr}(A_1, A_4) \geq 5$, $\mathrm{cr}(A_2, A_3) \geq 5$, $\mathrm{cr}(A_2, A_4) \geq 5$ and $\mathrm{cr}(A_3, A_4) \geq 4$. Moreover, by a discussion of possible subdrawings, we can verify that $\mathrm{cr}(A_3, A_4) \geq 5$. Clearly, also $\mathrm{cr}(A_k, A_k) \geq 6$ holds for any $k = 1, \dots, 4$. Thus, all lower-bounds of number of crossing of configurations from $\mathcal{M}$ are summarized in Tab. 3.

2.2 Main results

Lemma 1 *Let D be a good drawing of $G+D_n$, $n>2$, in which $\mathrm{cr}_D(T^i,T^j)\neq 0$ for any different subgraphs T^i and T^j. Let $2|R_D|+|S_D|>2n-2\left\lfloor \frac{n}{2}\right\rfloor$ and let $T^n, T^{n-1}\in R_D$ be different subgraphs with $\mathrm{cr}_D(T^n\cup T^{n-1})\geq 4$. If both conditions*

$$\mathrm{cr}_D(G\cup T^n\cup T^{n-1},T^i)\geq 10 \qquad \text{for any } T^i\in R_D\setminus\{T^n,T^{n-1}\}, \tag{2}$$

$$\mathrm{cr}_D(G\cup T^n\cup T^{n-1},T^i)\geq 7 \qquad \text{for any } T^i\in S_D \tag{3}$$

hold, then there are at least $6\left\lfloor \frac{n}{2}\right\rfloor\left\lfloor \frac{n-1}{2}\right\rfloor+2\left\lfloor \frac{n}{2}\right\rfloor$ crossings in D.

*P*roof. We denote by $r=|R_D|$ and $s=|S_D|$. By the assumption of lemma, any $T^i\notin R_D\cup S_D$ satisfies the condition $\mathrm{cr}_D(G\cup T^n\cup T^{n-1},T^i)\geq 4$, and the number of T^i that cross the graph G at least two times is equal to $n-r-s$. By fixing of the graph $G\cup T^n\cup T^{n-1}$ we have

$$\mathrm{cr}_D(G+D_n)=\mathrm{cr}_D(K_{6,n-2})+\mathrm{cr}_D(K_{6,n-2},G\cup T^n\cup T^{n-1})+\mathrm{cr}_D(G\cup T^n\cup T^{n-1})\geq$$

$$\geq 6\left\lfloor \frac{n-2}{2}\right\rfloor\left\lfloor \frac{n-3}{2}\right\rfloor+10(r-2)+7s+4(n-r-s)+4=6\left\lfloor \frac{n-2}{2}\right\rfloor\left\lfloor \frac{n-3}{2}\right\rfloor+6r+3s+4n-16\geq$$

$$\geq 6\left\lfloor \frac{n-2}{2}\right\rfloor\left\lfloor \frac{n-3}{2}\right\rfloor+3\Big(2n-2\left\lfloor \frac{n}{2}\right\rfloor+1\Big)+4n-16\geq 6\left\lfloor \frac{n}{2}\right\rfloor\left\lfloor \frac{n-1}{2}\right\rfloor+2\left\lfloor \frac{n}{2}\right\rfloor.$$

This completes the proof. □

Lemma 2 *Let D be a good drawing of $G+D_n$ with the vertex notations of the graph G as in Fig. 1(a), $n>2$. If $T^n\in R_D$ such that F^n has configuration $A_i\in\mathcal{M}_D$, for $i=1,3,4$, then*

$$\mathrm{cr}_D(T^n,T^k)\geq 3 \qquad \text{for any } T^k\in S_D. \tag{4}$$

*P*roof. Let, in D, the graph F^n has configuration A_1. If $T^k\in S_D$ with $\mathrm{cr}_D(T^n,T^k)=2$, then the vertex t_k must be placed in a region with at least three vertices of G on its boundary, see Fig. 2. Since $T^k\in S_D$, the vertex t_k cannot be placed in the region bounded by 4-cycle of the graph G. Moreover, if t_k is placed in another regions, then $\mathrm{cr}_D(F^n,T^k)>3$. The same idea can be used for configurations A_3 and A_4. This completes the proof. □

Remark that the property (4) is not true for configuration A_2, see the proof of the following statement.

Collorary 1 *Let D be a good drawing of $G+D_n$ with the vertex notations of the graph G as in Fig. 1(a), $n>2$, in which $\mathrm{cr}_D(T^i,T^j)\neq 0$ for any different subgraphs T^i and T^j. If $T^n,T^{n-1}\in R_D$ such that F^n,F^{n-1} have configurations A_1, A_2, respectively, then*

$$\mathrm{cr}_D(G\cup T^n\cup T^{n-1},T^k)\geq 7 \qquad \text{for any } T^k\in S_D. \tag{5}$$

*P*roof. Let, in D, the graphs F^n,F^{n-1} have configurations A_1,A_2, respectively. The configurations A_1 and A_2 are represented by the cyclic permutations $P_{85}=(125643)$ and $P_8=(132546)$, respectively.

- If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^{n-1}, T^k) = 2$, then the vertex t_k must be placed in the region with four vertices of G and one vertex t_{n-1} on its boundary, see Fig. 2. Thus, the graph $F^k = G \cup T^k$ can be represented only by two possible cyclic permutations $P_{81} = (126453)$ and $P_{95} = (156423)$. By the above mentioned algorithm we have

$$d_{COG}(''P_{26}'', ''P_{85}'') = d_{COG}(''P_{99}'', ''P_{85}'') = 4,$$

where $\overline{P_{81}} = (135462) = P_{99}$ and $\overline{P_{95}} = (132465) = P_{26}$. By the properties of the cyclic permutations we have $\mathrm{cr}_D(T^n, T^k) \geq 4$. Thus, $\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^k) \geq 1 + 4 + 2 = 7$.

- If $\mathrm{cr}_D(T^{n-1}, T^k) \geq 3$ for any subgraph $T^k \in S_D$, then $\mathrm{cr}_D(G \cup T^n \cup T^{n-1}, T^k) \geq 1+3+3 = 7$.

□

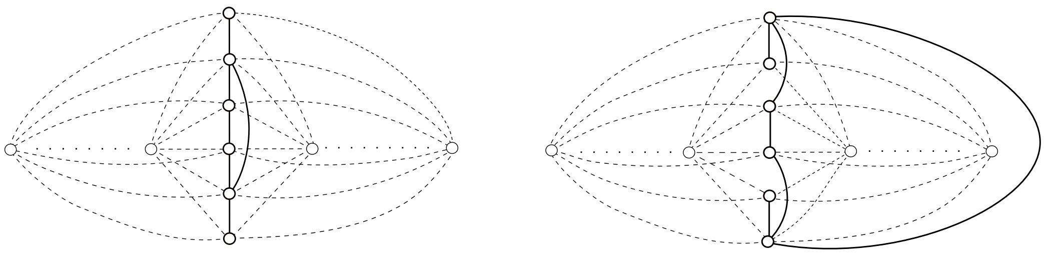


Fig. 3. Two good drawings of $G + D_n$.

Theorem 1 $\mathrm{cr}(G + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$.

*P*roof. In Fig. 3 there are the drawings of $G + D_n$ with $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus, $\mathrm{cr}(G + D_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$. We prove the reverse inequality by induction on n. The graph $G + D_1$ is planar, hence $\mathrm{cr}(G + D_1) = 0$. It is clear from Fig. 1(b) that $\mathrm{cr}(G + D_2) \leq 2$. The graph $G + D_2$ contains a subdivision of $K_{3,4}$, and therefore $\mathrm{cr}(G + D_2) \geq 2$. So, $\mathrm{cr}(G + D_2) = 2$ and the result is true for $n = 1$ and $n = 2$.

Suppose now that, for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + D_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor, \tag{6}$$

and let

$$\mathrm{cr}(G + D_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor \qquad \text{for any } m < n. \tag{7}$$

The drawing D has the following property:

$$\mathrm{cr}_D(T^i, T^j) \neq 0 \qquad \text{for all } i, j = 1, 2, \ldots, n,\ i \neq j. \tag{8}$$

To prove it assume that there are two different subgraphs T^i and T^j such that $\mathrm{cr}_D(T^i, T^j) = 0$. Without loss of generality let $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. One can easy to verify that $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 2$.

As $\mathrm{cr}(K_{6,3}) = 6$, we have $\mathrm{cr}_D(T^k, T^{n-1} \cup T^n) \geq 6$ for $k = 1, 2, \dots, n-2$. So, for the number of crossings in D holds

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D\left(G + D_{n-2}\right)+\mathrm{cr}_D(T^{n-1}\cup T^n)+\mathrm{cr}_D(K_{6,n-2}, T^{n-1}\cup T^n)+\mathrm{cr}_D(G, T^{n-1}\cup T^n) \geq$$

$$\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 2\left\lfloor\frac{n-2}{2}\right\rfloor + 6(n-2) + 2 = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.$$

This contradicts (6), and therefore $\mathrm{cr}_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2, \dots, n$, $i \neq j$. Our assumption on D together with $\mathrm{cr}(K_{6,n}) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{6,n}) < 2\left\lfloor\frac{n}{2}\right\rfloor.$$

Hence, if we denote $r = |R_D|$ and $s = |S_D|$, then

$$0r + 1s + 2(n-r-s) < 2\left\lfloor\frac{n}{2}\right\rfloor.$$

Thus, $r \geq 1$ and $2r + s > 2n - 2\left\lfloor\frac{n}{2}\right\rfloor$. We will fix one or two subgraphs with a contradiction with the assumption that there are less than $6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor$ crossings in the following cases:

Case 1: $\mathrm{cr}_D(G) = 0$.

We will deal with the sets of configurations $\{A_1, A_2\}$ in the drawing D.

1) $\{A_1, A_2\} \not\subseteq \mathcal{M}_D$.

   a) Let $A_2 \notin \mathcal{M}_D$ and $A_i \in \mathcal{M}_D$ for some $i \in \{1, 3, 4\}$, or let $A_2 \in \mathcal{M}_D$ and $A_i \in \mathcal{M}_D$ for some $i \in \{3, 4\}$. Without lost of generality, we can assume that $T^n \in R_D$ with F^n having configuration A_i. Thus, by fixing of the graph F^n using Lemma 2 we have

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D(K_{6,n-1})+\mathrm{cr}_D(K_{6,n-1}, G\cup T^n)+\mathrm{cr}_D(G\cup T^n) \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+$$

$$+5(r-1)+4s+3(n-r-s) = 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+2r+s+3n-5 \geq 6\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+$$

$$+2n-2\left\lfloor\frac{n}{2}\right\rfloor + 1 + 3n - 5 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.$$

   b) Let $\mathcal{M}_D = \{A_2\}$ and, without lost of generality, let $T^n \in R_D$.
   If there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) = 2$, then we fix the graph F^n having configuration A_2 and we obtain the same inequalities as in the previous case. So, assume that there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^n, T^k) = 2$. We can easily verify that $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 6 + 2 = 8$ for any $T^i \in R_D$, because both F^n and F^i have configuration A_2. Similarly by a discussion for two possible drawings of the graph T^k, see the proof of Corollary 1, we can verify that $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 7$ for any $T^i \in S_D$ and $\mathrm{cr}_D(G \cup T^n \cup T^k, T^i) \geq 6$ for any $T^i \notin R_D \cup S_D$. Thus, by fixing of the graph $G \cup T^n \cup T^k$ we have

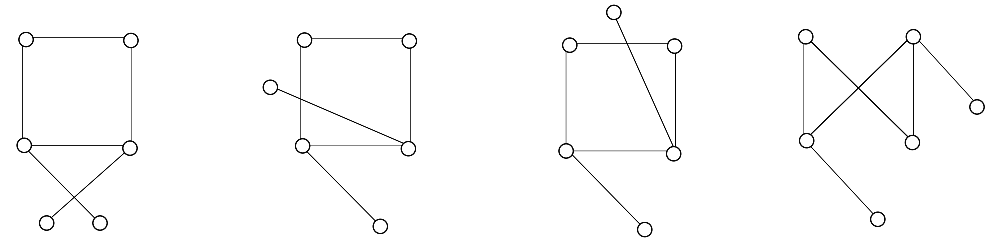
$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G \cup T^n \cup T^k) + \mathrm{cr}_D(G \cup T^n \cup T^k) \geq$$

$$\geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 8(r-1) + 7s + 6(n-r-s) + 3 = 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 2r + s+$$

$$+6n - 12 \geq 6\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 2n - 2\left\lfloor\frac{n}{2}\right\rfloor + 1 + 6n - 12 \geq 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor.$$

2) $\{A_1, A_2\} \subseteq \mathcal{M}_D$.
Without lost of generality let us fix any two $T^n,\ T^{n-1} \in R_D$ such that $F^n,\ F^{n-1}$ have configurations A_1, A_2, respectively. Then condition (2) is true by Tab. 3 and condition (3) holds by Corollary 1. Thus, all assumption of Lemma 1 are fulfilled.

Case 2: $\mathrm{cr}_D(G) = 1$.



(a) (b) (c) (d)

Fig. 4. Four possible drawings of the graph G with one crossing among its edges.

Since $r \geq 1$, without lost of generality we assume $T^n \in R_D$. In all four possible drawing of the graph G it is possible to verify that $\mathrm{cr}_D(G \cup T^n, T^i) \geq 4$ for any subgraph T^i, $i = 1, \ldots, n-1$. Thus, by fixing of the graph F^n we obtain

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n) \geq$$

$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Case 3: $\mathrm{cr}_D(G) \geq 2$.

We are able to use the same idea as in the previous case for all possible drawing of the graph G with a possibility of an existence of a subgraph $T^i \in R_D$ in the considering drawing D.

This completes the proof of the main theorem. □

2.3 Corollaries

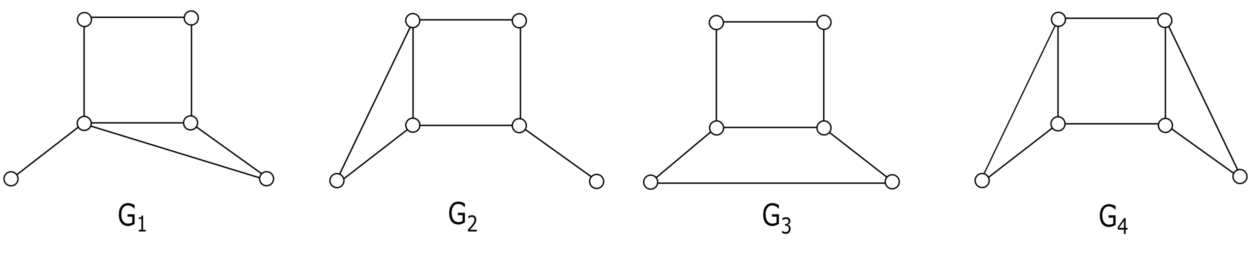


Fig. 5. Four graphs G_1, G_2, G_3, and G_4 by adding new edges to the graph G.

In Fig. 2 we are able to add some edges to the graph G without another crossings. So the drawing of the graphs $G_1 + D_n$, $G_2 + D_n$, $G_3 + D_n$, and $G_4 + D_n$ with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings is obtained. Thus, the next results are obvious.

Collorary 2 $\mathrm{cr}(G_i + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$, *where* $i = 1, \ldots, 4$.

Remark that the crossing numbers of the graphs $G_3 + D_n$ and $G_4 + D_n$ were obtained in [8], [5] without using the vertex rotation.

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