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SECONDARY MATHEMATICS MISCONCEPTION AS A MAIN OBSTACLE IN SOLVING HIGHER MATHEMATICS PROBLEMS

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Abstract. Paper deals with the most common misconception in secondary mathematics and its influence on solving tasks of higher mathematics. We provide several examples from our pedagogical experiences, main misconceptions connected to the mentioned examples and our solutions for possible re-education of mentioned misconceptions.

Keywords: concept of function, mathematics analysis, basic mathematics misconceptions

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1 Introduction

Mathematics knowledge of university students seems to be lower and lower year by year. They struggle with many misconceptions concerning secondary level mathematics. These misconceptions cause freshman's poor performance in university mathematical courses.
As long as the lack of the secondary level mathematical concepts and misconceptions about basic mathematical objects, concepts and operations continue, it is impossible to learn and understand new concepts on university level.
It is well known that students can master various formalized rules or computational algorithms without a deeper understanding of the issue. This fact leads students to formal and incomplete knowledge. Students are able to learn some techniques to solve limits, derivatives, integrals, but there may be differences between their concept image[1] of the considered notion and formal definition of this mathematical notion. Such reduced and restricted understanding of the content of fundamental mathematical concepts can cause the inability to understand the proofs of some (often basic) mathematical theorems. Formal knowledge is a major obstacle to solving the nonstandard problems of higher mathematics, requiring a deep comprehension of the concept, not only routine algorithmic skills.

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[1] The set of all the mental pictures associated in the student´s mind with the concept name, together with all the properties characterizing them. [1]

All the mentioned is manifested especially in mathematical analysis courses. The limit process, derivatives, integrals, all the "epsilon - delta" definitions and their applications are used without a concept understanding. Formal knowledge is enlarged by every other mathematical concept that is learned. It is a pity, because without a good mathematical background there are no good engineers, economists, computer scientists, and representatives of applied sciences.

We distinguish between random mistake (it may be the result of faulty memory, cognitive overload etc.) which can be corrected by a little bit warning and misconception. There are some definitions of misconception in current literature. According to [5] misconceptions are the wrong concepts or conceptions that a person assumes it as true and uses as a habit. In [4] is misconception defined as the inconsistency between the concept that we want students to learn and the mental model that they build in their minds.
We understand the misconception about mathematical concept in sense of [7] as a set of mathematical objects considered by students to be examples of the concept which is not necessarily the same as the set of mathematical object determined by the definition.

2 Function concept misconceptions

The concept of function is probably one of the most important concepts in contemporarily mathematics; it is included in the content of analysis (calculus), algebra, and geometry. Therefore, this concept is also important in secondary school mathematics and current curriculum emphasizes the importance of functions. In spite of that, a lot of freshman university students obtained the misconceptions about this concept.
The emergence of function concept can be dating to the end of 17th century, to the beginnings of infinitesimal calculus. Many mathematicians deal with the notion of function in their researche (for example, Newton (1642-1727) was one of the first mathematicians to show how functions could be developed in infinite power series, Leibniz (1646-1716) was the first, who used the term "function" in 1673, etc.) and the definition of this notion has changed over time. Jean Bernoulli in 1718 defined a function of a variable as a quantity that is composed in some way from that variable and constant. His student, Leonard Euler, later changed the term quantity by the term analytical expression. Finally, Dirichlet separated the concept of notion from its analytical representation and defined function as an arbitrary correspondence between two variables, so that to any value of the independent variable it is associated one and only one value of the dependent variable. The modern concept of functions is based on the set theory (in the literature also referred to as Dirichlet – Bourbaki concept) – the function is a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). Instead of the term correspondence, one can talk about set of ordered pairs that satisfies a certain condition.
Current definition includes some correspondences that are not recognized as a function by mathematicians of previous two centuries. For example: discontinuous functions, function defined on split domain (by different rules or formulas) on different subdomains (Leonard Euler did not consider this correspondence as function) or functions by means of a graph.

The formal definition of function is introduced in Slovakia to high school students. In spite of that many high school and university students can memorize definition of the function in the form mentioned above, but their understanding of this notion is restricted the same way as it has been in history of mathematics and concepts of previous generations of mathematicians.

A similar opinion is given by Walton, who claims that many students make comparable definitions of function but they are not aware of what these definitions represent [8]. Another study set forth that the students cannot understand the concepts that are featured especially within operations in analysis courses [2].

During our pedagogical practise we often meet the students whose concept of the function is built on the formula conception (like Euler). The parallel of phylogeny and ontogenesis of the concept of function is very obvious in this case. It can be caused by examples of functions used in high school textbooks to illustrate this concept. These examples are often (sometimes exclusively) functions whose rule of correspondence is given by a formula. This fact can lead to students´ misconception or formulating the incorrect definition like “A function is a formula, an algebraic expression, etc” or “It is an equation expressing a certain relation between two objects”.

Especially, we can observe the biggest problem with non-polynomial functions. Students are able to understand somehow the expression $y = x^2 + 5x + 6$. Problems with calculating and solving particular tasks start with function with “alphabetical name” like logarithm, sinus, cosine etc. Let’s have a look on some examples of students operations with these functions:

Example 2.1: Logarithmic functions:

a) $\ln x(\ln x - 1) = 0 \;\Rightarrow\; x = \dfrac{\ln x - 1}{\ln}$

b) $\ln x - \ln a = \ln(x - a)$

Example 2.2: Goniometric functions

a) $\sin(\sin x) = \sin^2 x$

b) $\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = x + \dfrac{\cos x}{\sin x}$

As we can see, the problem is very serious and its remedy is not easy. It will take quite a lot of time to make changes in students' conceptual maps. We cannot be sure about the results of our re-educating process.

3 Secondary mathematics misconceptions in mathematical analysis

We will provide brief summary of the most crucial problems based on secondary mathematics misconceptions and its influence on solving problems of mathematical analysis (or calculus).

3.1 Limits process

It is quite an easy algorithm to calculate limit of sequence or a function. The main problem is misunderstanding of the concept of function on secondary school. There are few examples of students’ work which can demonstrate this situation (see also example 2.1 and 2.2):

Example 3.1: $\lim\limits_{n\to\infty} \dfrac{2^n - 1}{2^n + 1} \bullet \dfrac{\log_2}{\log_2} = \lim\limits_{n\to\infty} \dfrac{\log_2 2^n - \log_2 1}{\log_2 2^n + \log_2 1} = \lim\limits_{n\to\infty} \dfrac{n-0}{n-0} = 1$

Example 3.2: $\lim\limits_{x\to 0}\frac{\sin(x+2\pi)}{\sin(x+3\pi)}=\frac{x+2\pi}{x+3\pi}=\frac{2}{3}$

On these two examples we can observe the wrong understanding of the function mentioned in paragraph 2. Not only mentioned students have no idea what the specific mathematical notation means. We think that the main problems are:

- Misconception of the function concept
- Insufficient training of calculation with other than polynomial functions
- Students' opinion – everything is on Wikipedia, why we should learn something?

Problems like these are more and more common in our lessons of mathematics. It might take a lot of time and energy to demonstrate the problem with such calculations, to argue why specific ways of solving is not correct and the necessity to re-educate all the misconceptions. The results are not pleasing. To correct so strong misconception of function on University level is almost impossible.

Example 3.3: $\lim\limits_{n\to\infty}\frac{(n+1)^n}{n^n}=\lim\limits_{n\to\infty}\frac{n^n+2n+1^n}{n^n}$

This example is very typical misunderstanding of basic algebra identities. Once they finally can correctly expand $(a+b)^2$ we can find out very formal knowledge of it: "there is always $2ab$ middle term". If we asked them about binomial theorem, they know about it and some of them start to think about how it could be connected to our specific problem (and when teacher asked about it, it has to be connected). Counting with fraction is well known problem in school mathematics. Unfortunately, this problem survives very long time. Let's have a look on next two examples:

Example 3.4: $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\left(\frac{\tan x}{x^2}-\frac{\sin x}{x}\right)$

Example 3.5: $\lim\limits_{n\to\infty}\left(\frac{1}{\alpha+1}-\frac{1}{n+\alpha+1}\right)=\lim\limits_{n\to\infty}\left(\frac{1}{\alpha+1}-\frac{1}{n}-\frac{1}{\alpha+1}\right)=\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right)=0$

Misconception concerning fractions is also very common and questions concerning "when we can stroke out some numbers" or "why it was possible now and not before" sounds incredible on the university students of (applied) mathematics (yes, we are talking about students of mathematics). Problems with calculation with fractions are very frequent in calculations of integrals (see paragraph 3.3).

3.2 Derivative of function

We think that there is no easier algorithm for solving problems in mathematics analysis than finding the derivatives of functions. It is such routine that everyone should handle it. By everyone we mean someone who already has managed concept of function and operations with them (addition, multiplication, composing). If there is missing this essential knowledge we can find several kinds of problems:

Example 3.6: $\left(5\sin^2\left(\cos 3x\right)\right)' = 5\cos^2 x - \sin x + 3$

Example 3.7: $\left(\dfrac{\sin x + x^2}{5^x}\right)' = \dfrac{\cos x + 2x}{x.5^{x-1}}$

We have to say that students are allowed to use the derivative rules on the test. Example 6 shows us misconception of derivative of compound function. Example 7 represents the most common mistake – quotient rule for derivative and again misconception of the function. Students often do not recognize polynomial functions from exponential ones.

3.3 Integral and primitive function

To master integration of the function is much harder than master derivatives or other concept of mathematical analysis. There are not such easy rules. Students have to make sense for using correct substitution or method of solving. Besides that, there is still the biggest obstacle in solving the integrals - "lower mathematics" – fractions and concept of functions.

Example 3.8: $\int_0^{\pi} \sin^2 x\,dx = \int_0^{\pi} \dfrac{1}{\left(\sin^2 x\right)^{-1}}\,dx = \left[\left(-\text{cotg } x\right)^{-1}\right]_0^{\pi}$

Example 3.9:

a) $\int t^2 \dfrac{t^2}{t^3+1}\,dt = \int t^2 dt - \int \dfrac{t^2}{t^3+1}\,dt$

b) $\int \dfrac{1}{t^2+3t-1}\,dt = \int \dfrac{1}{t^2}\,dt + \int \dfrac{1}{3t}\,dt + \int (-1)\,dt$

c) $\int \dfrac{dx}{\sin x + \cos x + 5} = \int \dfrac{dx}{\sin x} + \int \dfrac{dx}{\cos x} + \int \dfrac{dx}{5}$

d) $\int \dfrac{dx}{3x^2+2x+2} = \dfrac{1}{3}\int \dfrac{dx}{x^2+2x+2}$

Some of these problems remain unchanged since the first semester of university studies. Students seem to be not able to connect knowledge from algebra (division by x means multiplication by 1 over x in some contexts). As we mentioned in Section 3.1 of Function Limits, the misconception of basic algebraic identities and function concepts is a problem leading to an inappropriate solution to higher mathematical problems. In parallel with higher mathematics, the exaltation of secondary mathematics appears in lessons. Nevertheless their knowledge of basic schemes and concepts are very formal and our effort to change it has very weak effect.

4 Suggested solutions and their results

Our experiences as university mathematics teachers show that students need longer time period to reconsider their conceptual schemas but they do not have so much time. They did have it on secondary school. Unfortunately, this time has been wasted. Some of the students find out that mathematics is not good study program for them and they leave the study. The rest still remain and suffer.

We have to handle with this situation as good as possible. We'd like to encourage students to learn harder on their chosen study program. Therefore we started to prepare better environment for obtaining necessary mathematical skills:

- 2 lessons per week from mathematical analysis (voluntary subject),
- Enlarged consultation time for the groups,
- Using ICT on mathematical analysis lessons:
  - Demonstration tool (teachers work),
  - Tool for modelling, observation and practising (students work),
  - Preparing e-learning support for the subject,
- Preparing special voluntary courses focused on secondary mathematics.

The results are not very encouraging. The more we try the less the students try. We tried to help them more than students wanted. Therefore, this academic year we stopped some activities and remain only voluntary lessons and e-learning support. The observed results so far are:

- Students started searching more information (there are only very basic information in e-learning course)
- Many students realized their weaknesses and they started to ask more on lessons or they ask to have more consultation time

5 Conclusion

We have attempted to summarize the most important misconceptions we have encountered in our pedagogical practice. We think many university mathematics teachers have seen the same or very similar mistakes. According to the constructivist approach and its rules, the teacher should perceive the error as the developmental stage of the pupil's understanding of mathematics and the impulse for further work [4]. Unfortunately, some teachers of mathematics are not sure how to use error as an impulse for further work. Therefore we think that to solve problem of misconceptions (not only of function concept) we should do the following:

- To educate future mathematics teachers how to identify misconceptions and suggest an effective teaching strategy that helps their students to develop a correct understanding of these concepts.
- To encourage in-service teachers to be interested to take into consideration the misconceptions about basic mathematical concepts and implement an effective teaching strategy that helps their students to develop a correct understanding of these concepts.
- To consider 4 years of bachelor study (some of the technical universities already have it) or to implement into accreditation materials obligatory subject "The foundation of mathematics" focused on the most critical parts of mathematics on primary and secondary school.

Since, every learned concept in mathematics is closely related with previous or upcoming concepts, a difficulty experienced in learning a certain concept causes difficulties in learning many further concepts and paves the way for misconceptions (Altun, In: [1]). Therefore, building correct conceptual maps is necessary for further education.

References

[1] CANSIZ, S., KŰCŰK, B, ISLEYEN, T. Identifying the secondary school student´s misconceptions about functions. *Procedia Social and Behavioral Sciences*, vol. 15, 2011: ISSN 1877-0428

[2] GRAVEMEIJER, K., DOORMAN, M. Context Problems in Realistic Mathematics Education: A Calculus Course as an Example. *Educational Studies in Mathematics*, Vol. 39, 1999: pp. 111-129, ISSN 0013-1954

[3] KLEIN, F. *Elementary mathematics from an advanced standpoint*. (E. R. Hedrick & C. A. Noble, trans.). New York, NY: Dover, 1945. (Original work published 1908)

[4] MICHAEL, J. Misconceptions – What Students Think They Know, *Advanced in Physiology Education*, Vol. 26, No. 1, 2002: pp. 4 -6, ISSN 1043-4046

[5] ÖZKAN, C. M. Misconceptions in radicals in high school mathematics. *Procedia Social and Behavioral Sciences*, vol. 15, 2011: pp. 120 – 127, ISSN 1877-0428

[6] STEHLÍKOVÁ, N., CACHOVÁ, J. Konstruktivistické přístupy k vyučování a praxe. Studijní materiály k projektu Operační program Rozvoj lidských zdrojů), JČMF, Praha, 2006

[7] VINNER, S., DREYFUS, T. Images and definitions for the concept of function. *Journal for Research in Mathematical Education*, Vol. 20, No. 4, 1989: pp. 356 – 366, ISSN 00218251

[8] WALTON, K. D., Examining Functions in Mathematics and Science Using Computer Interfacing. *School Science and Mathematics*, Vol. 88, 1988: pp. 604-609, ISSN 1949-8594

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