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MODELS USED IN FUZZY LINEAR REGRESSION

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Abstract. The paper summarizes fuzzy linear models that were introduced in fuzzy regression analysis. The exhaustive review is extended with new models. This contribution is intended as a guide for implementing known fuzzy regression methods. However, it can be used as an inspiration for researchers developing new approaches.

Keywords: fuzzy regression, possibilistic regression, fuzzy linear model, fuzzy observations

Mathematics subject classification: Primary 62E86, 03E72.

Fuzzy linear regression analysis is a vital alternative to commonly used statistics-based regression methods. In the fuzzy linear regression analysis, a wide variety of fuzzy linear models can be used for approximation of a linear dependence, according to a set of observations. To orient the user in selection of an appropriate model, we issue an exhaustive review of fuzzy linear models that were expected while developing model parameter estimators. We provided the models with references on parameter estimators. We also introduced new fuzzy linear models that can be used in fuzzy regression. We classified the models into three categories according to datatype they are intended for. We also provided remarks on model features from the perspective of model predictions. The review is intended as a guide for implementing fuzzy regression estimators as well as an inspiration for researchers developing new parameter estimation methods aimed at fuzzy data.

1 Introduction

Fuzzy regression analysis is an alternative to the commonly used statistical regression analysis. Both in fuzzy regression analysis and in statistical regression analysis, a regression model is used to describe a functional relationship between a dependent y and independent variables x. Parameters of the model are estimated utilizing a set of observations of the variables x and y. With estimated parameters, the regression model can be used for prediction of the dependent variable value for a specific combination of the independent variable values. While any deviation of the prediction from the corresponding observation is supposed to be due to a random error or measurement errors in the statistical regression, the fuzzy regression expects the deviations due to fuzziness. The fuzziness can be inherently given by the nature of observations (e.g. observations described by linguistic terms) [31, 27, 24]; however, it can also be due to the imprecise observed data or the indefiniteness of the system structure and parameters [3]. The fuzzy regression is also a viable alternative to statistical regression when the dataset is insufficient to support statistical regression analysis [20].

Probably the most commonly used model in statistical regression analysis is a linear model. In fuzzy regression analysis, the linear dependence of y on x is also most often expected. However, the fuzzy regression analysis offers several fuzzy models that can be used to approximate the expected dependence. The observations of the dependent and independent variables can be either real value numbers x, y or fuzzy numbers $\tilde{X}, \tilde{Y}$. In this paper, we bring an exhaustive review of fuzzy models that were expected during development of model parameter estimators. We provide the models with references on parameter estimators. We also present new fuzzy linear models that can be used in fuzzy regression. The referred fuzzy models are classified into three categories according to the type of observations they are intended for. We further provide remarks on model features from a perspective of model predictions.

2 Preliminaries

This section originates from [2, 36].

2.1 Fuzzy numbers

Definition 1. A *fuzzy set* A defined on a universe X of elements x is defined by a mapping $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is the membership degree of x to the fuzzy set A, $\mu_A(x) = 1$ means full membership of x in A, and $\mu_A(x) = 0$ expresses non-membership. If a fuzzy set has a positive degree of membership to a single element $x \in X$, then the set is called *fuzzy singleton*.

Definition 2. Let A be a fuzzy set defined on the universe X of elements x. The set of elements that belong to the fuzzy set A at least to a degree α is called *α-level set* and it is given as

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\},$$

where $0 < \alpha \leq 1$. The α-level set of A for $\alpha = 1$ is called a *core* of the fuzzy set A.

Definition 3. A fuzzy subset $\tilde{A}$ defined on $\mathbb{R}$ with membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ is called a *fuzzy number* if

(a) $\mu_{\tilde{A}}$ is normal, i.e. $\exists x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x_0) = 1$,
(b) $\mu_{\tilde{A}}$ is fuzzy convex, (i.e. $\mu_{\tilde{A}}(tx + (1-t)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}, \forall t \in [0,1], x, y \in \mathbb{R}$),
(c) $\mu_{\tilde{A}}$ is upper semi-continuous on $\mathbb{R}$ (i.e. $\forall \varepsilon > 0 \exists \delta > 0$: $\mu_{\tilde{A}}(x) - \mu_{\tilde{A}}(x_0) < \varepsilon, |x - x_0| < \delta$),
(d) $\mu_{\tilde{A}}$ is compactly supported, i.e. $\mathrm{cl}\{x \in \mathbb{R}; \mu_{\tilde{A}}(x) > 0\}$ is compact, where $\mathrm{cl}(A)$ denotes the closure of the set A.

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the space of fuzzy numbers.

Remark. *A real number $a \in \mathbb{R}$ is also a fuzzy number $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ which has the positive degree of membership only for the element a, i.e. the fuzzy number $\tilde{A}$ is the fuzzy singleton. Such numbers are called crisp numbers.*

Definition 4. Let L and R be continuous decreasing functions $L, R : [0, +\infty) \to [0,1]$ fulfilling $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, invertible on $[0,1]$. Moreover, let $m^L_{\tilde{A}}, m^R_{\tilde{A}} \in \mathbb{R}$, where

$m^L_{\tilde{A}} \leq m^R_{\tilde{A}}$, and $\alpha_{\tilde{A}}, \beta_{\tilde{A}} \in \mathbb{R}^+$; then a fuzzy set $\tilde{A}$ is said to be an *L-R type fuzzy number*, if its membership function is

$$\mu_{\tilde{A}} = \begin{cases} L\left(\frac{m^L_{\tilde{A}} - x}{\alpha_{\tilde{A}}}\right), & x < m^L_{\tilde{A}}, \\ 1, & m^L_{\tilde{A}} \leq x \leq m^R_{\tilde{A}}, \\ R\left(\frac{x - m^R_{\tilde{A}}}{\beta_{\tilde{A}}}\right), & x > m^R_{\tilde{A}}, \end{cases}$$

where $m^L_{\tilde{A}}$ and $m^R_{\tilde{A}}$ are the *left* and the *right points of the core*, and $\alpha_{\tilde{A}}$ and $\beta_{\tilde{A}}$ are the *left* and the *right spread* of the fuzzy number $\tilde{A}$. The L-R type fuzzy number $\tilde{A}$ can be written as

$$\tilde{A} = \left(m^L_{\tilde{A}}, m^R_{\tilde{A}}, \alpha_{\tilde{A}}, \beta_{\tilde{A}}\right)_{LR}.$$

2.2 Fuzzy algebra

Using the extension principle [34], fuzzy algebra can be developed for the L-R type fuzzy numbers.

Definition 5. The r-*level set* of the fuzzy number $\tilde{A}$ is defined as $\tilde{A}_r = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq r\}$. The set $\tilde{A}_r$ is a closed interval $\tilde{A}_r = [\tilde{A}^-_r, \tilde{A}^+_r]$.

Definition 6. *Scalar multiplication* $(\lambda \cdot \tilde{A})$ is multiplication between the real number $\lambda \in \mathbb{R}$ and the fuzzy number $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ where $(\lambda \cdot \tilde{A}) \in \mathbb{R}_{\mathcal{F}}$. This operation is defined as

$$(\lambda \cdot \tilde{A})_r = \left\{\lambda x \middle| x \in \tilde{A}_r\right\} = \lambda \tilde{A}_r, \forall r \in [0, 1].$$

Note that $\lambda \tilde{A}_r$ is the usual product of a number and a subset of $\mathbb{R}$. For any $\lambda, \kappa \in \mathbb{R}$ and $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$, it holds that

$$(\lambda\kappa) \cdot \tilde{A} = \lambda \cdot (\kappa \cdot \tilde{A}).$$

Remark. *It holds that* $((\lambda \cdot \tilde{A})^+_r - (\lambda \cdot \tilde{A})^-_r) \geq (\tilde{A}^+_r - \tilde{A}^-_r)$ *for* $|\lambda| \geq 1, \forall r \in [0, 1]$*; and* $((\lambda \cdot \tilde{A})^+_r - (\lambda \cdot \tilde{A})^-_r) < (\tilde{A}^+_r - \tilde{A}^-_r)$ *for* $|\lambda| < 1, \forall r \in [0, 1]$.

Definition 7. Sum of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{A} \oplus \tilde{B}$, where $\tilde{A}, \tilde{B}, (\tilde{A} \oplus \tilde{B}) \in \mathbb{R}_{\mathcal{F}}$, is given as

$$(\tilde{A} \oplus \tilde{B})_r = \left\{x + y \middle| x \in \tilde{A}, y \in \tilde{B}\right\} = \tilde{A}_r + \tilde{B}_r, \forall r \in [0, 1].$$

(a) The sum of fuzzy numbers is commutative and associative, i.e.

$$\tilde{A} \oplus \tilde{B} = \tilde{B} \oplus \tilde{A}$$

and

$$\tilde{A} \oplus (\tilde{B} \oplus \tilde{C}) = (\tilde{A} \oplus \tilde{B}) \oplus \tilde{C}, \forall \tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}_{\mathcal{F}}.$$

(b) The fuzzy singleton $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ (i.e. $\mu_{\tilde{0}}(x) = 1$ for $x = 0$ and $\mu_{\tilde{0}}(x) = 0$ for $x \neq 0$) is the neutral element with respect to (w.r.t.) $\oplus$, i.e.

$$\tilde{A} \oplus \tilde{0} = \tilde{0} \oplus \tilde{A} = \tilde{A}$$

for any $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$.

(c) None of $\tilde{A} \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has an opposite in $\mathbb{R}_{\mathcal{F}}$ (w.r.t. $\oplus$).

(d) For any $\lambda, \kappa \in \mathbb{R}$ with $(\lambda\kappa) \geq 0$ and any $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$, we have the distributive law

$$(\lambda + \kappa) \cdot \tilde{A} = \lambda \cdot \tilde{A} \oplus \kappa \cdot \tilde{A}.$$

This property does not hold for any $\lambda, \kappa \in \mathbb{R}$.

(e) For any $\lambda \in \mathbb{R}$ and $\tilde{A}, \tilde{B} \in \mathbb{R}_{\mathcal{F}}$, the distributive law is fulfilled:

$$\lambda \cdot (\tilde{A} \oplus \tilde{B}) = \lambda \cdot \tilde{A} \oplus \lambda \cdot \tilde{B}.$$

Remark. *$\tilde{A}_r + \tilde{B}_r$ is the sum of two intervals (as subsets of $\mathbb{R}$). It means that* $((\tilde{A} \oplus \tilde{B})^+_r - (\tilde{A} \oplus \tilde{B})^-_r) \geq \max\{(\tilde{A}^+_r - \tilde{A}^-_r), (\tilde{B}^+_r - \tilde{B}^-_r)\}, \forall r \in [0, 1]$.

Definition 8. Product of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{A} \otimes \tilde{B}$, where $\tilde{A}, \tilde{B}, (\tilde{A} \otimes \tilde{B}) \in \mathbb{R}_{\mathcal{F}}$, is defined as

$$(\tilde{A} \otimes \tilde{B})^-_r = \min\left\{\tilde{A}^-_r \tilde{B}^-_r, \tilde{A}^-_r \tilde{B}^+_r, \tilde{A}^+_r \tilde{B}^-_r, \tilde{A}^+_r \tilde{B}^+_r\right\},$$

and

$$(\tilde{A} \otimes \tilde{B})^+_r = \max\left\{\tilde{A}^-_r \tilde{B}^-_r, \tilde{A}^-_r \tilde{B}^+_r, \tilde{A}^+_r \tilde{B}^-_r, \tilde{A}^+_r \tilde{B}^+_r\right\}, \forall r \in [0, 1].$$

(a) The fuzzy singleton set $\tilde{1} \in \mathbb{R}_{\mathcal{F}}$ (i.e. $\mu_{\tilde{1}}(x) = 1$ for $x = 1$ and $\mu_{\tilde{1}}(x) = 0$ for $x \neq 1$) is the neutral element w.r.t. $\otimes$, i.e.

$$\tilde{A} \otimes \tilde{1} = \tilde{1} \otimes \tilde{A} = \tilde{A},$$

for any $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$.

(b) None of $\tilde{A} \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$ has an opposite in $\mathbb{R}_{\mathcal{F}}$ (w.r.t. $\otimes$).

(c) For any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}_{\mathcal{F}}$ we have

$$((\tilde{A} \oplus \tilde{B}) \otimes \tilde{C})_r \subseteq (\tilde{A} \otimes \tilde{C})_r + (\tilde{B} \otimes \tilde{C})_r, \forall r \in [0, 1].$$

and, in general, distributivity does not hold.

(d) For any $\tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{R}_{\mathcal{F}}$, where none of the supports of $\tilde{A}, \tilde{B}, \tilde{C}$ contain 0, we have

$$\tilde{A} \otimes (\tilde{B} \otimes \tilde{C}) = (\tilde{A} \otimes \tilde{B}) \otimes \tilde{C}.$$

3 Fuzzy linear models

Fuzzy linear models describe linear relations between the independent variables x and the dependent variable y. The models differ in expected datatypes of the variables x, y, in datatypes of model parameters, as well as in number of the parameters. Since the model selection is primarily datatype driven, we used the datatype as the feature for classification of the models.

3.1 Models for fuzzy dependent and fuzzy independent variables

Fuzzy models belonging to this category, expect that observations of the dependent y, as well as observations of all m independent variables x are fuzzy numbers, i.e. $\tilde{X}_i, \tilde{Y} \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{1, \ldots, m\}$. The extensively used model of this category [9, 29, 28, 10, 15, 23, 30, 18, 19, 21] is given as

$$\tilde{Y} = \tilde{A}_0 \oplus \left(\tilde{A}_1 \otimes \tilde{X}_1\right) \oplus \ldots \oplus \left(\tilde{A}_m \otimes \tilde{X}_m\right), \tag{1}$$

where $\tilde{A}$ denotes fuzzy parameters of the regression model, and $\tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{0, \ldots, m\}$.

Other fuzzy linear models of this category can be inferred from the model (1) through specifying some model parameters as real value numbers. In our search, we found three such models that were used in development of parameter estimators. The first model expects that all parameters, except the intercept, are real value numbers a (i.e. $a_i \in \mathbb{R}, \forall i \in \{1, \ldots, m\}$ and $\tilde{A}_0 \in \mathbb{R}_{\mathcal{F}}$). The model is given as

$$\tilde{Y} = \tilde{A}_0 \oplus a_1 \cdot \tilde{X}_1 \oplus \ldots \oplus a_m \cdot \tilde{X}_m. \tag{2}$$

Parameter estimators for this model were introduced in [8, 32, 1].

The second model considers all parameters as real value numbers (i.e. $a_i \in \mathbb{R}, \forall i \in \{0, \ldots, m\}$). The model is given as

$$\tilde{Y} = a_0 \oplus a_1 \cdot \tilde{X}_1 \oplus \ldots \oplus a_m \cdot \tilde{X}_m. \tag{3}$$

Parameter estimators for this model were introduced in [22, 1, 14, 25].

The third model represents an extension of the fuzzy linear model (3) due to a fuzzy error term $\tilde{\delta}$. The model is given as

$$\tilde{Y} = a_0 \oplus a_1 \cdot \tilde{X}_1 \oplus \ldots \oplus a_m \cdot \tilde{X}_m \oplus \tilde{\delta}, \tag{4}$$

where $a_i \in \mathbb{R}, \forall i \in \{0, \ldots, m\}$ and $\tilde{\delta} \in \mathbb{R}_{\mathcal{F}}$. Parameter estimators for this model were introduced in [16, 17, 5, 7, 6, 18].

The idea to extend fuzzy linear models with the fuzzy error term $\tilde{\delta}$ can be applied to models (1) and (2). By adding the fuzzy error term, new fuzzy linear models can be obtained. They are given as

$$\tilde{Y} = \tilde{A}_0 \oplus \left(\tilde{A}_1 \otimes \tilde{X}_1\right) \oplus \ldots \oplus \left(\tilde{A}_m \otimes \tilde{X}_m\right) \oplus \tilde{\delta}, \tag{5}$$

where $\tilde{\delta}, \tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{0, \ldots, m\}$; and

$$\tilde{Y} = \tilde{A}_0 \oplus a_1 \cdot \tilde{X}_1 \oplus \ldots \oplus a_m \cdot \tilde{X}_m \oplus \tilde{\delta}, \tag{6}$$

where $\tilde{\delta}, \tilde{A}_0 \in \mathbb{R}_{\mathcal{F}}$ and $a_i \in \mathbb{R}, \forall i \in \{1, \ldots, m\}$. Note that these two models were not considered yet by developers of the parameter estimators. Considering [7], we hypothesize that adding the fuzzy error term might improve precision of the fuzzy linear regression.

3.2 Models for fuzzy dependent and real value independent variables

Fuzzy models belonging to this category expect that observations of all independent variables x are real value numbers ($x_i \in \mathbb{R}, \forall i \in \{1, \ldots, m\}$), and observations of the dependent variable y are fuzzy numbers $\tilde{Y} \in \mathbb{R}_{\mathcal{F}}$. For such data, most of parameter estimators [12, 26, 4, 8, 33, 11, 15, 7, 1, 13, 18, 35] expect the linear dependence described by the model

$$\tilde{Y} = \tilde{A}_0 \oplus \tilde{A}_1 \cdot x_1 \oplus \ldots \oplus \tilde{A}_m \cdot x_m, \tag{7}$$

where $\tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{0, \ldots, m\}$.

Derived from the model (7), new models can be obtained when some model parameters would be real value numbers. Such a way, a parallel of the model (2) can be obtained. The new fuzzy linear model is given as

$$\tilde{Y} = \tilde{A}_0 \oplus \left(a_1 x_1 + \ldots + a_m x_m\right), \tag{8}$$

where $a_i \in \mathbb{R}, \forall i \in \{1, \ldots, m\}$ and $\tilde{A}_0 \in \mathbb{R}_{\mathcal{F}}$. According to our search, no parameter estimator is available for this model yet.

Parameter estimator published in [7] considered an extension of the model (7) with the fuzzy error term $\tilde{\delta}$. The extended model is given as

$$\tilde{Y} = \tilde{A}_0 \oplus \tilde{A}_1 \cdot x_1 \oplus \ldots \oplus \tilde{A}_m \cdot x_m \oplus \tilde{\delta}, \tag{9}$$

where $\tilde{\delta}, \tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{0, \ldots, m\}$.

The extended model (9) can be simplified, when all parameters of the regression model are considered as real value numbers, i.e.

$$\tilde{Y} = (a_0 + a_1 x_1 + \ldots + a_m x_m) \oplus \tilde{\delta}, \tag{10}$$

where $\tilde{\delta} \in \mathbb{R}_{\mathcal{F}}$, and $a_i \in \mathbb{R}, \forall i \in \{0, 1, \ldots, m\}$. An estimator based on this model was presented in [18].

3.3 Models for real value dependent and fuzzy independent variables

The opposite of the previous category are fuzzy models that are intended for datasets where the observations of the independent variables x are fuzzy numbers ($\tilde{X}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{1, \ldots, m\}$), while the observation of the dependent variable y are real value numbers $y \in \mathbb{R}$. To ensure the requirement on the model output y, all parameters related to the fuzzy independent variables $\tilde{X}$ must be also fuzzy numbers. The basic model of this category is given as

$$y = \tilde{A}_0 \oplus \left(\tilde{A}_1 \otimes \tilde{X}_1\right) \oplus \ldots \oplus \left(\tilde{A}_m \otimes \tilde{X}_m\right), \tag{11}$$

where $\tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{0, \ldots, m\}$. A parameter estimator for this model was presented in [10].

A new model can be obtained with the intercept represented as a real value number. Thus, the model is given as

$$y = a_0 \oplus \left(\tilde{A}_1 \otimes \tilde{X}_1\right) \oplus \ldots \oplus \left(\tilde{A}_m \otimes \tilde{X}_m\right), \tag{12}$$

where $a_0 \in \mathbb{R}$, and $\tilde{A}_i \in \mathbb{R}_{\mathcal{F}}, \forall i \in \{1, \ldots, m\}$. As far as we know, no parameter estimator was designed for this model.

4 Model attributes

Since each real value number can be expressed as a fuzzy number, we can say that the fuzzy linear model (5), respectively the model (1), are the general fuzzy linear models, and the remaining models are their simplifications. However, existence of the simplified fuzzy linear models is well founded. The simplified models have naturally implemented assumptions on model prediction in their structures. Let us demonstrate the importance of this fact on several examples.

Significance of prior assumptions on model predictions is best evident in models for real value dependent and fuzzy independent variables. Models of this class are intended for applications where real value predictions of the models are required. Parameter estimators based on such models provide parameters that guarantee the desired model predictions, while parameter estimators based on the general model (1) do not.

Another good example are models for fuzzy dependent and real value independent variables. Predictions of these models are fuzzy numbers $\tilde{Y} = \left(m^L_{\tilde{Y}}, m^R_{\tilde{Y}}, \alpha_{\tilde{Y}}, \beta_{\tilde{Y}}\right)_{LR}$, where $m^L_{\tilde{Y}}, m^R_{\tilde{Y}}$ are left and right cores of the predictions $\tilde{Y}$, and $\alpha_{\tilde{Y}}, \beta_{\tilde{Y}}$ are their left and right spreads. The fuzzy linear models (7) and (9) are known for dependence of the model prediction spreads $\alpha_{\tilde{Y}}$ and $\beta_{\tilde{Y}}$ on absolute values of the independent variables x [26, 7]. The fuzzy linear models (8) and (10), which belong to the same class, have different properties. The spreads $\alpha_{\tilde{Y}}$ and $\beta_{\tilde{Y}}$ are independent of x for these models [18]. Thus, the models (7) and (9) allow to model relationships where the prediction spreads are dependent on x, while the models (8) and (10) are appropriate for relations with no correlation of the spreads and x.

In this context, limitations of the models (7) and (9), arising from the essence of the fuzzy arithmetic, should be mentioned. As stated above, the model prediction spreads $\alpha_{\tilde{Y}}$ and $\beta_{\tilde{Y}}$ are dependent on x for these two models. For positive values of x, the spreads increase with increasing values of x. For negative values of x, the spreads decrease with increasing values of x. This limitation can be overcome by translating the observations along x [3]. Using axes translations, decreasing spreads $\alpha_{\tilde{Y}}, \beta_{\tilde{Y}}$ with increasing values of x for $x \in \mathbb{R}^+$; and increasing spreads $\alpha_{\tilde{Y}}, \beta_{\tilde{Y}}$ with decreasing values of x for $x \in \mathbb{R}^-$ can be achieved with these two models.

5 Conclusion

Here we provided user guidelines for selection of fuzzy linear models. From the wide variety of fuzzy linear models, the appropriate model for a given dataset should be selected according to datatype of the observations. However, the spreads of model predictions should respect the nature of the approximated relation, and this fact should be reflected in the model selection as well. We hope that the review will provide an inspiration for researchers developing new parameter estimators. To support further development of the fuzzy regression, we proposed four new fuzzy linear models. However, other new models can be introduced for fuzzy dependent and fuzzy independent variables, and for fuzzy dependent and real value independent variables. Considering some model parameters to be real value numbers, new fuzzy linear models can be formulated for these two model categories.

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