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SINUSOID CHARACTERISTICS IN APPLETS

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Abstract. The paper deals with visualisation of sinusoid characteristics. It discusses the importance of visualisation and interactive manipulations in the process of understanding the heart of the matter as the essential stage to acquire higher levels of knowledge and skills. Combining different teaching and learning aids and methods helps not only to supply but also develop various styles of cognition.

Keywords: visualisation, mathematical competencies, sinusoid characteristic

Mathematics Subject Classification: Primary 42B05; Secondary 33B10, 42B05

1 Introduction

In mathematical courses curriculum, there are some entries, which make a clue for substantial technical phenomena. They enable to describe them and even control through engineering processes. Such are sinusoids that are indisputably widely used, mostly for depiction of signals, distributed periodically, within various media. One can speak about electromagnetic waves, e.g. radio or TV waves; various radiations, e.g. light; mechanical waves – propagation of sound, corn waves, or water surface waves; not leaving out vibration of rotors or other mechanical systems; or one can speak even about gravitational waves, seismic waves or others. From mathematical point of view, signals can be generally expressed as real functions $f(t)$. Being elementary orthogonal functions they satisfy the conditions of the simplest mathematical approximation model $f(t)=\sum_k a_k \vartheta_k(t)$ with superposition property providing signal decomposition into $a_k \vartheta_k(t)$ parts. The response of system to the applied input function can be then explored by individual responses of parts, producing the net response in the sum. The study of vibrations usually begins with the key chapter on periodic signals that are articulated by means of Fourier series of harmonic motions. Harmonic motion as a function of time is sinusoidal; it is characterized by period T, angular frequency Ω, amplitude A and phase φ what allows to transform it into a function of frequency and analyse the signal by means of amplitude and phase spectra. Since relatively large variety of its possible analytical formulations, the understanding of each element in goniometric function representation is very important and leads to the success in the study and following engineering practice.

2 Sinusoids – Ambiguity and Pitfalls

2.1 Ambiguity Problem

Mathematically, sinusoids are the graphs of sine or cosine functions. As they have the same shape but a shift from the origin of the Cartesian coordinate system in basic position, it leads to more possibilities of their anallytical expression and gives rise to become a stumbling block for students. For instance the sinusoid in the figure 1 can be expressed as

$$\text{a) } 2\sin(t-\frac{\pi}{4})\text{; b) } 2\sin(t+(-\frac{\pi}{4}))\text{; c) } 2\cos(t-\frac{3\pi}{4})\text{; d) } 2\cos(t+(-\frac{3\pi}{4})),$$

where the value of the curve shift is restricted to the interval $\langle -\pi, \pi \rangle$. Moreover, taking into account the orthogonality of (cos t, sin t) system, it can be also expressed as $\sqrt{2}\sin(t)-\sqrt{2}\cos(t)$.

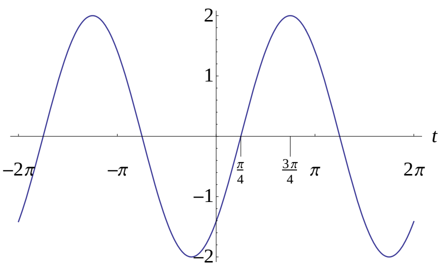

Fig. 1. Sinusoid

$$2\sin(t-\frac{\pi}{4}) = 2\sin(t+(-\frac{\pi}{4})) = 2\cos(t-\frac{3\pi}{4}) = 2\cos(t+(-\frac{3\pi}{4})) = \sqrt{2}\sin(t)-\sqrt{2}\cos(t)$$

One posssibility, how to eliminate the ambiguity is to prefer only one form. Nevertheless, each representation has its advantages, what greatly facilitates the work with curve, so nobody should want to get rid of it. For instance, sine representation has the intuitive non-confusable position of starting point [0,0], and the shift could be easily recognized. On the other hand, the cosine repesentation could stand for signals that start at full power, and moreover, they correspond to exponentional (complex) representation $a\cos(\omega t+\varphi) = c\,e^{\omega t} + \bar{c}\,e^{\omega t}$, where c and $\bar{c}$ are complex conjugates and φ is also the argument of complex number c in polar form.

Going to simple and for students more transparent difference, in representation $a\sin(t-\varphi)$, φ stands directly for intersection point of the sinusoid with axis t, while taking into account the representation $a\sin(t+\varphi)$, the intersection point has to have the opposite sign, $-\varphi$.

What can help is the fact, that the sine and cosine graphs are shifted by $\pi/2$. What complicates the situation is that although a student is concious of the shift, she/he has to be aware of the shift direction.

2.2 Applets

To reveal this situation we decided to prepare applets as learning aids for students to help them better understand the role of used coefficients in the formulas.

Usually freshmen come to study at technical universities with relatively good familiarity with two concepts of sine and cosine functions. First, it is a practical concept of side ratios in right-angled triangle, sometimes also demonstrated within the unit circle. The second one is the approach through a standard sinusoidal graph of periodical real function in one variable, however perceived isolated, not in relation to the first concept, and in more abstract way. In calculus course, students work with graphs, they learn to understand the meaning, and changes of the graph with respect to coefficients *a, b, c* in expressions $a\sin x$, $a\sin(x+c)$, $a\sin(bx)$ (the same for cosine). The most strange for students is to understand the coefficient *b* responsible for the length of periodicity interval. A problem also occurs when putting all coefficients together in $a\sin(bx+c)$, especially in connection to shift of the starting/reference point of sinusoid (fig.2).

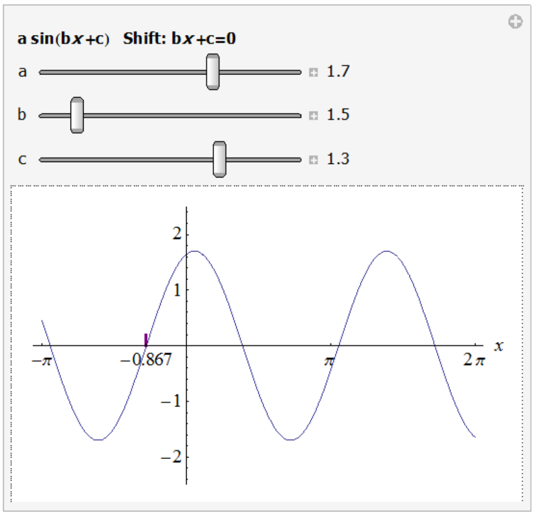

Fig. 2. Applet: $a\sin(bx+c)$, playing with coefficients.

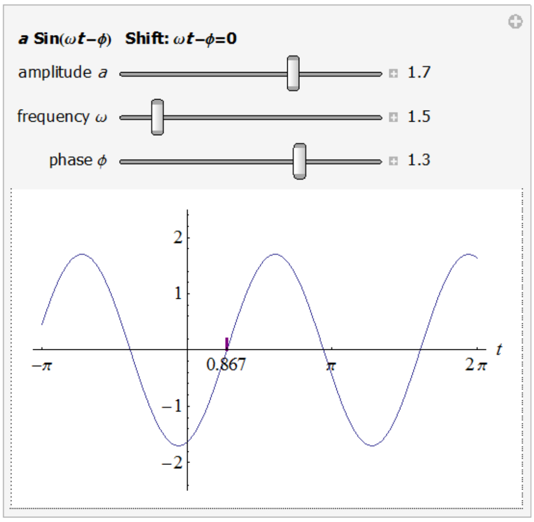

Fig. 3. Applet: harmonic signal $a\sin(\omega t-\varphi)$, reference point shift calculation.

Studying the elementary theory of signals, students of higher years can directly apply their previous skills. General coefficients *a*, *b*, *c* are substituted by technical terms: amplitude *a*, frequency ω, and phase φ (fig. 3).

Usually, there are no difficulties until the moment of harmonic analysis, when one has to decompose the periodic signal onto harmonic components, to calculate their frequencies, amplitudes and phases. The most problematic is calculation of phase, when a student is confronted with different formulas, even producing the different results. Alleviating the confusion, we have very good experience with kinematic way of sinusoid creation for sine and cosine representation, helping to achieve the necessary control. In presented applets, a periodical movement is represented by the movement along the unit circle. Simultauosly, the corresponding sinusoid is recorded in the Cartesian coordinate system. Guide angle *t* of the moving point is at each position transformed into the independent variable *t* of sine function. The value sin *t* is simply transferred horizontally from the vertical axis (fig. 3), while the value cos *t* is transferred vertically from the horizontal axis onto the vertical axis of the sinusoid graph (fig. 4). Recording the corresponding values we obtain desired sinusoids: sin *t*, and the rotated graph cos *t*, that can be then turned into its traditional position. For better recognition, the sine and cosine relations in the unit circle can be supported by image of the right angle triangle (seen in fig. 7)

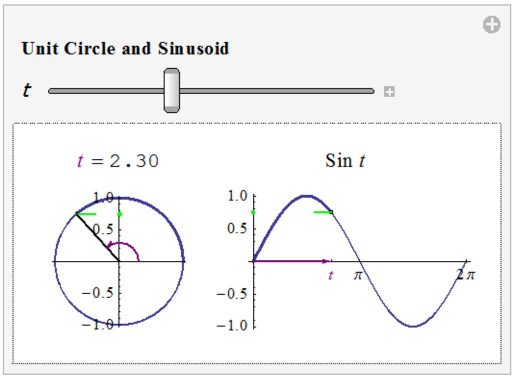

Fig. 4. Applet: sin *t*; kinematic approach.

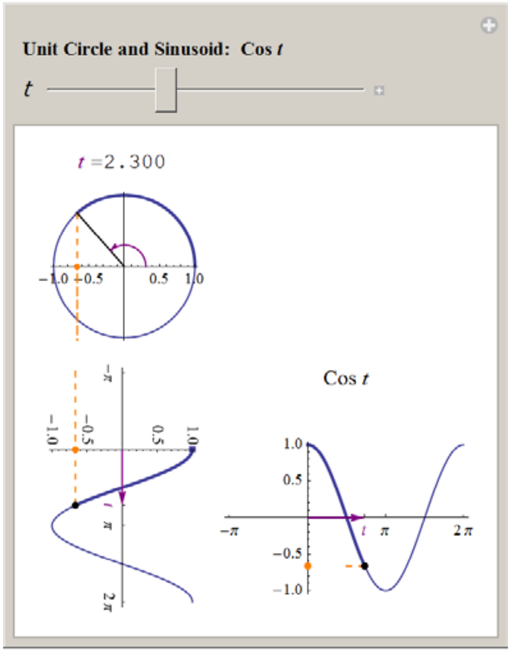

Fig. 5. Applet: cos *t*; kinematic approach.

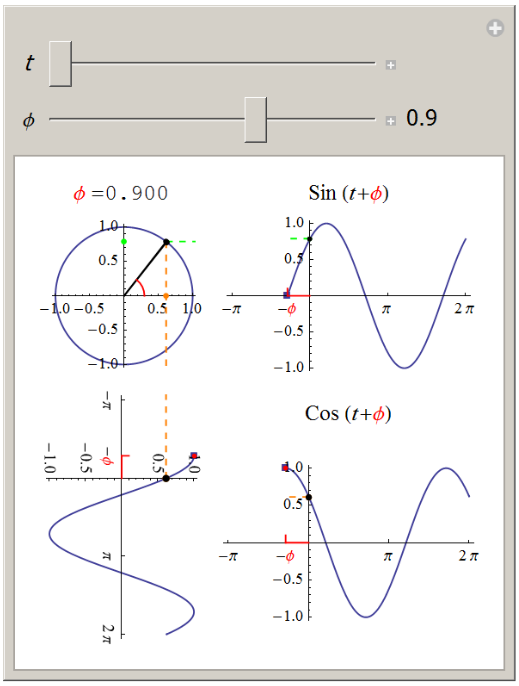

Fig. 6. Applet: $\sin(t+\varphi)$ & $\cos(t+\varphi)$,
one angle φ *2 different curves* $\sin\varphi$, $\cos\varphi$.
Innitial position of the movement.

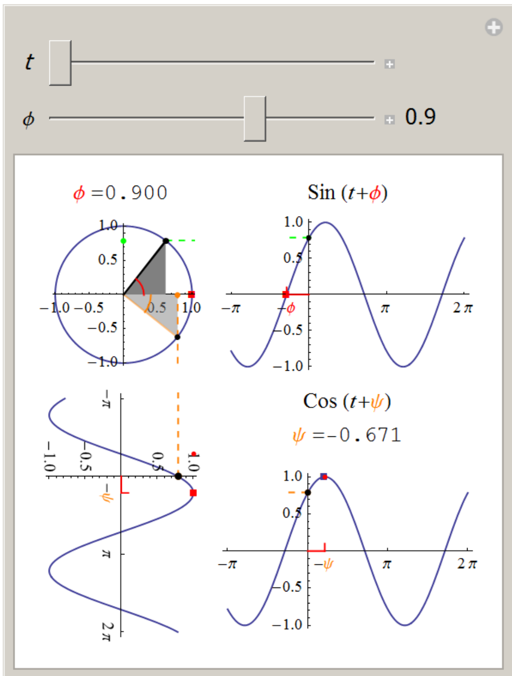

Fig. 7. Applet: $\sin(t+\varphi)$ & $\cos(t+\psi)$,
one curve 2 different angles φ, ψ.
Innitial position of the movement.

The shift of the sinusoidal initial reference point by the value φ can be demonstrated on the same principle. Starting the movement from the angle φ (fig. 6) on the unit circle produces two different curves, where the initial reference point on each curve is shifted by the same value φ or $-\varphi$ (depends on argument formulation). On the other hand, if we want to obtain the same sinusoid (fig. 7), it is necessary to start from different angle for each representation. As sine and cosine curves are shifted by $\pi / 2$, the initial values of angles differ also by $\pi / 2$.

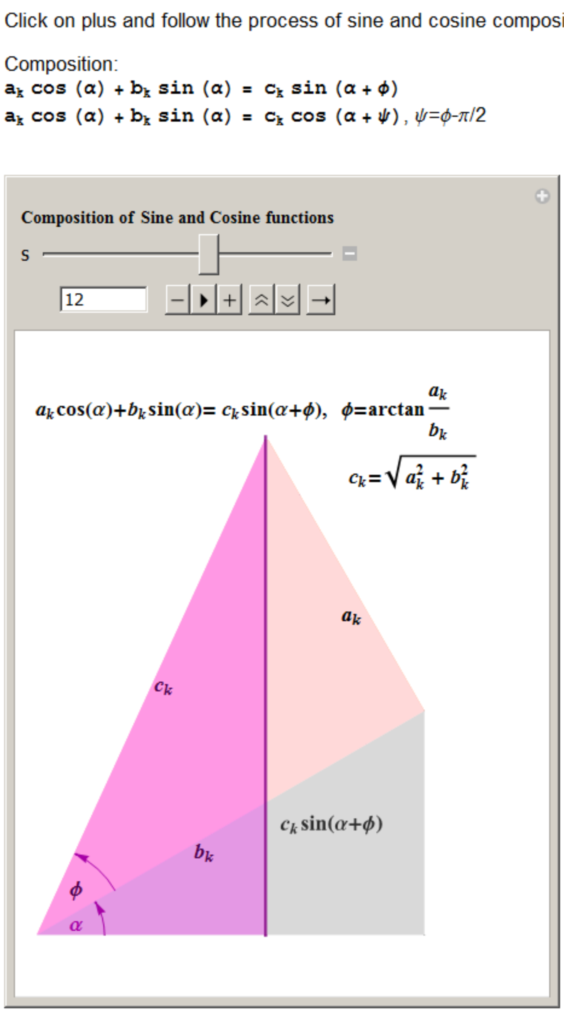

Fig. 8. Applet: Demonstration

$$c_k \sin(\alpha + \varphi) = a_k \cos\alpha + b_k \sin\alpha$$

$$c_k = \sqrt{a_k^2 + b_k^2}\,, \quad \tan\varphi = a_k / b_k\,.$$

While relations on composition of sine and cosine waves in trigonometric form of harmonic motion can be demonstrated by composition of right-angled triangles (fig. 8), phase counting itself holds another pitfall: from $\tan\varphi = a_k / b_k$ basically $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ($\varphi \in \left(-\frac{\pi}{2} + \mathrm{k}\pi, \frac{\pi}{2} + \mathrm{k}\pi\right), k \in N$), being the value of shifting sinusoid, basically $\varphi \in (-\pi, \pi)$, what always requires to consider the current part of the interval.

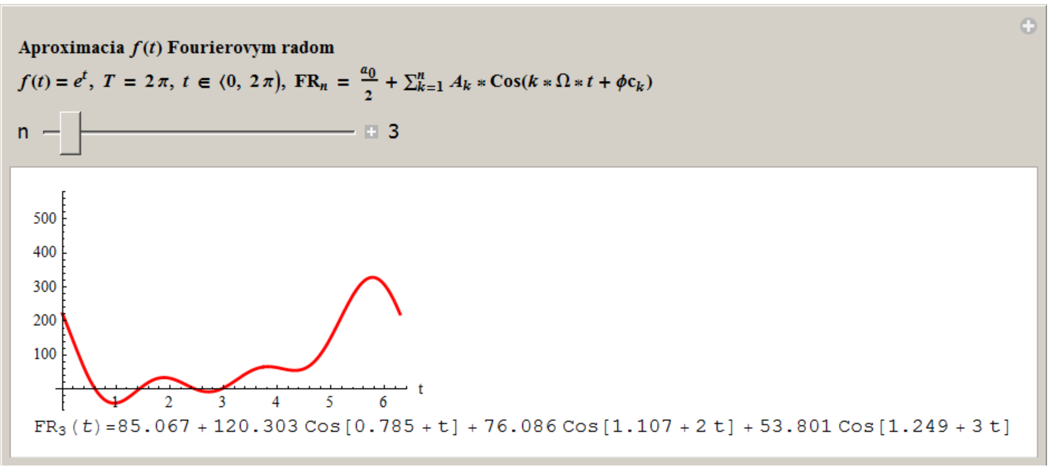

Fig. 9. Applet: Approximation *f* (*t*) with Fourier series. The sum of first 4 items.

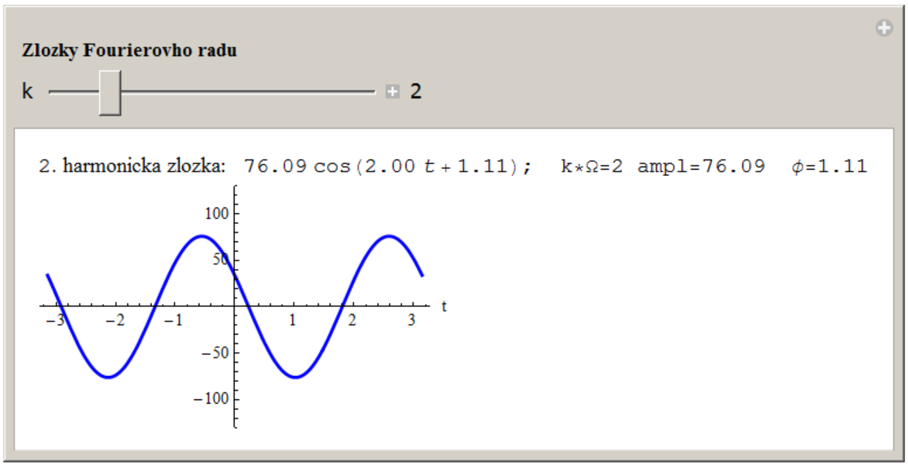

Fig. 10. Applet: Terms of Fourier series. Second harmonic and its characteristics.

3 Decisive Mathematical Competency

Elementary theory on signals is usually taught by technicians using technical textbooks, where mathematical details are rarely explained. Due to multiformity of analytical expressions, usage of formulas is not always consistent and frequently results in confusions. In this context, mathematical knowledge is inevitable. Presented applets can be highly conducive to deep comprehension. On the other side, engineering context directly motivates to study and gives actual background.

Nowadays, overwhelmed by modern technology, physical models have been gradually replaced by virtual teaching and learning aids, which allow to combine visual, read/(formulas) and manipulative approach, often equipped also by auditory output, attacking all senses not only supplying preferred learning styles but as well, involving and developing not preferred ones, with strong potential to affect the cognition. However, to aquire this, they do not work autonomly. Carefully formulated tasks, Socratic questioning, provoking to critical thinking, supplemented by necessary facts, actual background contribute to building mathematical competency part by part and step by step. However, working in passive mode is not sufficient.

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