Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

NUMERICAL SOLUTION OF A SPECIAL SINGULAR INTEGRAL EQUATION OF THE SECOND KIND

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Abstract. This paper shows the method for numerical solution of a Fredholm integral equation of the second kind which has kernel function with a special type of singularity. The numerical solution is based on a Nyström method with a singularity subtraction technique. It requires a singular integral to be computed. Here the singular integral is converted to a non-singular one and a standard numerical quadrature is used.

Keywords: integral equations, Nyström method, numerical quadrature, singular integral

Mathematics Subject Classification: Primary 65R20, 45B05; Secondary 65Z05

1 Introduction

Fredholm integral equations of the second kind describe many physical phenomena. A numerical solution of integral equations leads to a solution of a system of linear equations with fully populated matrices. Many of them are described in [1]. The speed of finding a numerical solution depends on the speed of filling the matrix of a system of linear equations. One way to find the numerical solution is based on approximating the integral by numerical quadrature. Such methods are called Nyström methods. Let us have an integral equation of the following form:

$$y(x) - \int_0^1 \frac{h(x,t)y(t)}{|x-t|^\alpha} dt = f(x), \tag{1}$$

where $h(x,t) \in \mathcal{C}[0,1]$ and $\alpha \in (0,1)$. Here we integrate a singular function. So the Nytröm method can not be used directly. The Nyström method with singularity subtraction will be described in this paper. Its idea is to weaken the singularity in the integral.

2 Singularity subtraction

First let us weaken the singularity of the kernel function as in [2].

$$\frac{h(x,t)y(t)}{|x-t|^\alpha} = \frac{h(x,t)[y(t)-y(x)]}{|x-t|^\alpha} + f(x)\frac{h(x,t)}{|x-t|^\alpha}. \tag{2}$$

Substituting (2) to (1) we get

$$y(x) - \int_0^1 \frac{h(x,t)[y(t)-y(x)]}{|x-t|^\alpha}dt - y(x)\int_0^1 \frac{h(x,t)}{|x-t|^\alpha}dt = f(x). \tag{3}$$

The first integrand on the left hand side of (3) is singular when $x = t$. To apply numerical quadrature, we need to approximate it by a finite function. One approximation can be

$$\frac{1}{|x-t|^\alpha} \approx s_\varepsilon(x,t) = \begin{cases} \frac{1}{|x-t|^\alpha}, |x-t| \geq \varepsilon \\ \frac{1}{\varepsilon^\alpha}, |x-t| < \varepsilon. \end{cases} \tag{4}$$

We get

$$y(x) - \int_0^1 h(x,t)[y(t)-y(x)]s_\varepsilon(x,t)dt - y(x)\int_0^1 \frac{h(x,t)}{|x-t|^\alpha}dt = f(x). \tag{5}$$

Now the first integral on the left hand side of (5) is not singular and we can use standard numerical quadrature. The problem is with the second integral. It is singular, but its values are finite. It can be converted to a non-singular one.

3 Application of the Nyström method

Now let us take a numerical quadrature

$$\int_0^1 g(t)dt \approx \sum_{i=0}^{n} \omega_i g(x_i). \tag{6}$$

Points $x_i \in [0,1]$ are called the node points and numbers ω_i are called weights of the numerical quadrature.

Now we can approximate the first integral in (5) by (6) and run x through the node points. We get a system of linear equations for a solution in the node points

$$y(x_i) - \sum_{j=0}^{n} \omega_i h(x_i,x_j)[y(x_j)-y(x_i)]s_\varepsilon(x_i,x_j) - y(x_i)\int_0^1 \frac{h(x_i,t)}{|x-t|^\alpha}dt = f(x_i). \tag{7}$$

For sufficiently small ε is (7) equivalent to

$$y(x_i) - \sum_{j=0, j\neq i}^{n} \omega_i h(x_i,x_j)\frac{[y(x_j)-y(x_i)]}{|x_i-x_j|^\alpha} - y(x_i)\int_0^1 \frac{h(x_i,t)}{|x-t|^\alpha}dt = f(x_i). \tag{8}$$

We can see that the exact formula for construction of approximation (4) is not needed. We only need finite approximation of $|x-t|^{-\alpha}$ outside a certain neighborhood of $x = t$.

4 Computation of improper integral

The last problem is to compute the last integral on the left hand side of (8).

$$I(x) = \int_0^1 \frac{h(x_i,t)}{|x-t|^\alpha} dt \tag{9}$$

It is an improper integral but it reaches finite values. We will use an action as in [3] for a similar function. Since $x > 0$ we can split the integral $I(x)$ into a sum of two integrals

$$I(x) = \int_0^1 \frac{h(x,t)}{|x-t|^\alpha} dt = \int_0^x \frac{h(x,t)}{(x-t)^\alpha} dt + \int_x^1 \frac{h(x,t)}{(t-x)^\alpha} dt. \tag{10}$$

Using substitution $u = x - t$ the left integral in (10) can be written as

$$\int_0^x \frac{h(x,t)}{(x-t)^\alpha} dt = \int_0^x \frac{h(x,x-u)}{u^\alpha} du \tag{11}$$

Using substitution $u = t - x$ the right integral in (10) can be written as

$$\int_0^x \frac{h(x,t)}{(x-t)^\alpha} dt = \int_0^{1-x} \frac{h(x,x+u)}{u^\alpha} du. \tag{12}$$

Finally using substitution $\tau = u^{1-\alpha}$, (11) can be written as a non-singular integral

$$\int_0^x \frac{h(x,x-u)}{u^\alpha} du = \frac{1}{1-\alpha} \int_0^{x^{1-\alpha}} h\left(x, x - \tau^{\frac{1}{1-\alpha}}\right) d\tau \tag{13}$$

and (12) can be written as a non-singular integral

$$\int_0^{1-x} \frac{h(x,x+u)}{u^\alpha} du = \frac{1}{1-\alpha} \int_0^{(1-x)^{1-\alpha}} h\left(x, x + \tau^{\frac{1}{1-\alpha}}\right) d\tau. \tag{14}$$

From (11), (12), (13) and (14) the integral $I(x)$ has the non-singular form

$$I(x) = \frac{1}{1-\alpha} \left[\int_0^{x^{1-\alpha}} h\left(x, x - \tau^{\frac{1}{1-\alpha}}\right) d\tau + \int_0^{(1-x)^{1-\alpha}} h\left(x, x + \tau^{\frac{1}{1-\alpha}}\right) d\tau. \right] \tag{15}$$

and the common numerical integration rule can be used.

5 Error estimation

By using methods of functional analysis the error estimate can be derived. The error has two factors. First is the factor of singular function $|x-t|^{-\alpha}$ and the second is the error on the numerical integration rule. All important facts are summarized by the following theorem. It is proved in [4].

Theorem 5.1. Let y be the solution of (1) with $h(x,t) \in \mathcal{C}([0,1] \times [0,1])$, and $\gamma \in (0,1)$. Let y_n be the solution of (8), where the quadrature is a compound midpoint rule. Then for a sufficiently large n, there exist constants c_0 and c_1 such that

$$\|y - y_n\| \le c_0 \left[\frac{2}{1-\gamma} \max_{x\in[0,1]} \omega\left(h_x, \frac{1}{n}\right) + \frac{2c_1}{n^{1-\gamma}} \right], \tag{16}$$

where ω is the modulus of continuity and $h_x(t) = y(t)h(x,t)$. Furthermore, if $y \in \mathcal{C}^2[0,1]$ and $h_x(t) \in \mathcal{C}^2([0,1] \times [0,1])$ for each $x \in [0,1]$, then for a sufficiently large n there exists c_2 such that

$$\|y - y_n\| \le \frac{c_2}{n^{2-\gamma}}. \tag{17}$$

For other quadrature rules, similar theorems can be proved. Singularity of the kernel function is a very important factor which affects error behavior. So the error behavior is not better for more precise numerical quadratures. So it is not necessary to take a higher order integration rule than $\mathcal{O}(n^{-2})$, where n is the number of the node points.

6 Numerical tests

In this section let us verify the error estimation, which is described by the theorem 5.1. Let us take the compound mid-point rule. It has error $\mathcal{O}(n^{-2})$. Let us choose $\alpha = \frac{1}{2}$. It is the most common singular factor because it is the Euclidean distance. Let us choose the right hand side function $f(x)$ such that the exact solution is e^x. The right hand side function $f(x)$ is computed symbolically by Maple [5]. The same software is used for solution of the system of linear equations (8).

Let us verify the error behavior for exact solution $y(x) = e^x$ and $h(x,t) = x+t$. The columns labeled "ratio" in following tables give the ratio of successive errors. Since $h_x \in \mathcal{C}^2([0,1])$ the column ratio is expected to be approximately $\sqrt{8} \approx 2.83$.

n	error	ratio	n	error	ratio
10	0.0124685	-	20	0.0050904	2.45
40	0.0016737	3.04	80	0.0004682	3.57
160	0.0001236	3.78	320	0.0000318	3.88

Tab. 1. Midpoint rule, exact solution $y(x) = e^x$, $h(x,t) = (x+t)$

As we can see in table 1, the result is consistent to the theory. From table 2 we can see that the error is not worse if we take the function $h(x,t) = \ln(x+t)$. This is despite the fact that that h_x does not have continuous derivatives when $x = 0$. The reason is that the error (16) is too pessimistic for only continuous functions.

n	error	ratio	n	error	ratio
10	0.0244284	-	20	0.0059047	4.14
40	0.0014633	4.04	80	0.0003745	3.91
160	0.0000992	3.78	320	0.0000272	3.64

Tab.2. Midpoint rule, exact solution $y(x) = e^x$, $h(x, t) = \ln(x + t)$

Now let us take the exact solution $y(x) = \sqrt{x}$. This function is Hölder continuous with constant $\frac{1}{2}$. So the ratio should be approximately $\sqrt{2} \approx 1.41$. Tables 3 and 4 show error behavior for the same functions $h(x, t)$.

n	error	ratio	n	error	ratio
10	0,0009572	-	20	0,0004938	1.94
40	0,0002767	1.78	80	0,0001400	1.98
160	0,0000583	2.40	320	0,0000239	2.44

Tab. 3. Midpoint rule, exact solution $y(x) = \sqrt{x}$, $h(x, t) = x + t$

n	error	ratio	n	error	ratio
10	0.0104430	-	20	0.0052149	2.00
40	0.0033060	1.57	80	0.0020864	1.58
160	0.0013055	1.60	320	0.0008075	1.61

Tab. 4. Midpoint rule, exact solution $y(x) = \sqrt{x}$, $h(x, t) = \ln(x + t)$

From tables 3 and 4 we can see, that the theorem is also consistent to the results. Now let use choose $\alpha = \frac{1}{4}$. The function $f(x)$ is chosen such that the exact solution is $y(x) = e^x$. The ratio should decrease to the value $\sqrt[4]{2^7} \approx 3.36$ for continuous $h(x, t) \in \mathcal{C}[0, 1]$. From tables 5 and 6 we see the consistency with the theory.

n	error	ratio	n	error	ratio
10	0.0054453	-	20	0.0013761	3.96
40	0.0003450	3.99	80	0.0000865	3.99
160	0.0000216	4.00	320	0.0000054	4.00

Tab. 5. Midpoint rule, exact solution $y(x) = e^x$, $h(x, t) = x + t$, $\alpha = \frac{1}{4}$

n	error	ratio	n	error	ratio
10	0.0036882	-	20	0.0009687	3.80
40	0.0002566	3.78	80	0.0000704	3.64
160	0.0000252	2.79	320	0.0000088	2.86

Tab. 6. Midpoint rule, exact solution $y(x) = e^x$, $h(x, t) = \ln(x + t)$, $\alpha = \frac{1}{4}$

7 Conclusion

For the solution of integral equations many methods have been developed. Each of them lead to the solution of a system of linear equations with fully populated matrices. Let us compare this method to piecewise linear collocation. The error is $\mathcal{O}(n^{-2})$ (see [1] for details) for continuous $h(x, t)$ and $y(x)$ $\in \mathcal{C}^2([0,\ 1])$. So the label ratio is expected to be 4. If we compare the column ratio in the tables above, we can see that we have very good results.

But this method has a big advantage. Piecewise linear collocation and other methods (for example product integration methods) have big computing time. The reason is that every element in the matrix of the system of linear equations is an integral which needs to be calculated. In case of the Nyström method with the singularity subtraction, only diagonal elements of the matrix are integrals. This fact rapidly decreases computing time.

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