Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH INTEGER ROOTS

POTŮČEK Radovan (CZ)

Abstract. This contribution is a follow-up and a completion to the previous author's papers dealing with the sums of the series of reciprocals of quadratic polynomials with non-zero roots. We summarize these results and deal with the sum of the series of reciprocals of the quadratic polynomials with zero roots. We derive the formulas for these sums and verify them by some examples evaluated using the computer algebra system Maple 16.

Keywords: Sum of the series, telescoping series, harmonic number, computer algebra system Maple

Mathematics subject classification: Primary 40A05; Secondary 65B10

1 Introduction and basic notions

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \,.$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum\limits_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \,.$$

The series $\sum\limits_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence of partial sums $\{s_n\}$ converges to s, i.e. $\lim\limits_{n\to\infty} s_n = s$. We say that the series $\sum\limits_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum\limits_{k=1}^{\infty} a_k = s$.

The *nth harmonic number* is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

H_0 being defined as 0. The *generalized harmonic numbers* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r},$$

where $H_{n,1} = H_n$ are harmonic numbers. Generalized harmonic number of order n in power 2 can be written as a function of harmonic numbers using formula (see [6])

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n}.$$

From formulas for $H_{n,r}$, where $r = 1, 2$ and $n = 1, 2, \ldots, 10$, we get the following Table 1:

n	1	2	3	4	5	6	7	8	9	10
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$	$\frac{771817}{352800}$	$\frac{1968329}{1270080}$

Tab. 1. Ten first harmonic numbers H_n and generalized harmonic numbers $H_{n,2}$.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum s of the reciprocals of all the nonzero triangular numbers

$$T_n = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2},$$

which create the sequence $\{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \ldots\}$, is

$$s = \sum_{n=1}^{\infty} \frac{1}{(n^2+n)/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2.$$

This can be shown by using the following sum of a telescoping series.

The *telescoping series* is a series whose partial sums eventually only have a fixed number of terms after cancellation. For example, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ simplifies as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \lim_{N\to\infty} \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \right] \\
&= \lim_{N\to\infty} \left[1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{N} + \frac{1}{N}\right) - \frac{1}{N+1} \right] \\
&= \lim_{N\to\infty} \left[1 - \frac{1}{N+1} \right] = 1.
\end{aligned}$$

2 The sum of the series of reciprocals of the quadratic polynomials with non-zero integer roots

In the papers [1], [2], [3], [4], and [5], the following results concerning the quadratic polynomials with non-zero integer roots were derived.

In the paper [1] it was derived that the sum $s(a,b)^{++}$ of the series $\sum_{\substack{k=1\\k\neq a,b}}^{\infty} \frac{1}{(k-a)(k-b)}$ of reciprocals of the quadratic polynomials with different positive integer roots a and b, $a < b$, is given by the formula

$$s(a,b)^{++} = \frac{1}{b-a}\left(H_{a-1} - H_{b-1} + 2H_{b-a} - 2H_{b-a-1}\right), \tag{1}$$

where H_n is the nth harmonic number.

In the paper [2] it was shown that the sum $s(a,b)^{--}$ of the series $\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)}$ of reciprocals of the quadratic polynomials with different negative integer roots a and b, $a < b$, is given by the formula

$$s(a,b)^{--} = \frac{1}{b-a}\left(H_{-a} - H_{-b}\right). \tag{2}$$

In the paper [3] we derived the sum $s(a,a)^{--}$ of the series $\sum_{k=1}^{\infty} \frac{1}{(k-a)^2}$ of reciprocals of the quadratic polynomials with double negative integer root a is given by the formula

$$s(a,a)^{--} = \frac{\pi^2}{6} - H_{-a,2}, \tag{3}$$

where $H_{-a,2}$ is the generalized harmonic number of order $-a$ in power 2.

The formula for the sum $s(a,a)^{++}$ of the series $\sum_{\substack{k=1\\k\neq a}}^{\infty} \frac{1}{(k-a)^2}$ of reciprocals of the quadratic polynomials with double positive integer root a was derived in the paper [4] and has the form

$$s(a,a)^{++} = \frac{\pi}{2} + H_{a-1,2}. \tag{4}$$

The sum $s(a,b)^{-+}$ of the series $\sum_{\substack{k=1\\k\neq b}}^{\infty} \frac{1}{(k-a)(k-b)}$ of reciprocals of the quadratic polynomials with integer roots $a < 0$ and $b > 0$ was derived in the paper [5] and is given by the formula

$$s(a,b)^{-+} = \frac{(b-a)\left(H_{-a} - H_{b-1}\right) + 1}{(b-a)^2}. \tag{5}$$

3 The sum of the series of reciprocals of the quadratic polynomials with zero roots

Now, we analyze three cases of zero roots of the quadratic polynomials – the case of two zero roots, the case of one zero and one negative integer root, and the case of one zero and one positive integer root. We shall use the following notation: $s(0,0)$ instead of $s(a,b)^{00}$, $s(a,0)$ for negative integer a instead of $s(a,b)^{-0}$, and $s(0,b)$ for positive integer b instead of $s(a,b)^{0+}$.

3.1 The sum of the series of reciprocals of the quadratic polynomials with two zero roots

A problem to determine the sum $s(0,0) = \sum_{k=1}^{\infty} \frac{1}{(k-0)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is so called *Basel problem.* This problem was posed by *Pietro Mengoli* (1625–1686) in 1644. In 1737 *Leonhard Euler* (1707–1783) showed his famous result

$$s(0,0) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} = 1.644\,934\,066\ldots\ .$$

This sum presents the value $\zeta(2)$ of the *Riemann zeta function* $\zeta(s) = \sum_{k=1}^{\infty} k^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. Note that the values of the nth partial sum

$$s_n(0,0) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

correspond to the values $H_{n,2}$, so their first ten values are presented in Table 1.

3.2 The sum of the series of reciprocals of the quadratic polynomials with one zero and one negative integer root

The sum $s(a,0)$ of the series of reciprocals of the quadratic polynomials with one zero and one negative integer root a, where $a = -A$, $A > 0$, whence $k - a = k + A$, is given by the following simple theorem:

Theorem 1 *Let $a = -A$ be a negative integer. Then it holds* $\sum_{k=1}^{\infty} \frac{1}{k(k+A)} = \frac{H_A}{A}$, *where H_A is the Ath harmonic number, so*

$$s(a,0) = \sum_{k=1}^{\infty} \frac{1}{k(k-a)} = \frac{H_{-a}}{-a}. \tag{6}$$

Proof : By using the partial fraction decomposition of the telescoping series above we get

$$\sum_{k=1}^{\infty} \frac{1}{k(k+A)} = \frac{1}{A} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+A} \right) = \frac{1}{A} \lim_{N\to\infty} \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+A} \right) =$$
$$= \frac{1}{A} \lim_{N\to\infty} \left[\left(1 - \frac{1}{1+A} \right) + \left(\frac{1}{2} - \frac{1}{2+A} \right) + \cdots + \left(\frac{1}{A} - \frac{1}{2A} \right) + \left(\frac{1}{A+1} - \frac{1}{2A+1} \right) + \cdots \right.$$
$$\left. \cdots + \left(\frac{1}{N-A-1} - \frac{1}{N-1} \right) + \left(\frac{1}{N-A} - \frac{1}{N} \right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+A} \right) \right].$$

After cancellation all of the inner terms starting from $\frac{1}{A+1}$ up to $\frac{1}{N}$ and considering the facts that $\lim_{N\to\infty} \frac{1}{\ell} = \frac{1}{\ell}$ and $\lim_{N\to\infty} \frac{1}{N+\ell} = 0$ for any positive integer ℓ we get

$$\sum_{k=1}^{\infty} \frac{1}{k(k+A)} = \frac{1}{A} \lim_{N\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{A} - \frac{1}{N+1} - \frac{1}{N+2} - \cdots - \frac{1}{N+A}\right) =$$
$$= \frac{1}{A}\left(1 + \frac{1}{2} + \cdots + \frac{1}{A}\right) = \frac{H_A}{A}.$$

3.3 The sum of the series of reciprocals of the quadratic polynomials with one zero and one positive integer root

The sum $s(0,b)$ of the series of reciprocals of the quadratic polynomials with one zero and one positive integer root b is given by the following simple theorem:

Theorem 2 *Let b be a positive integer. Then it holds*

$$s(0,b) = \sum_{\substack{k=1\\k\neq b}}^{\infty} \frac{1}{k(k-b)} = \frac{1 - bH_{b-1}}{b^2}. \tag{7}$$

The formula (7) *can be also written in the equivalent form* $s(0,b) = \frac{H_b - 2H_{b-1}}{b}$.

Proof : Obviously, by using the partial fraction decomposition, we can write

$$\sum_{\substack{k=1\\k\neq b}}^{\infty} \frac{1}{k(k-b)} = \sum_{k=1}^{b-1} \frac{1}{k(k-b)} + \sum_{k=b+1}^{\infty} \frac{1}{k(k-b)} =$$
$$= \frac{1}{b}\left[\sum_{k=1}^{b-1}\left(\frac{1}{k-b} - \frac{1}{k}\right) + \lim_{N\to\infty} \sum_{k=b+1}^{N}\left(\frac{1}{k-b} - \frac{1}{k}\right)\right] =$$
$$= \frac{1}{b}\left\{\left(\frac{1}{1-b} - 1\right) + \left(\frac{1}{2-b} - \frac{1}{2}\right) + \cdots + \left(-\frac{1}{2} - \frac{1}{b-2}\right) + \left(-1 - \frac{1}{b-1}\right) + \right.$$
$$+ \lim_{N\to\infty}\left[\left(1 - \frac{1}{b+1}\right) + \left(\frac{1}{2} - \frac{1}{b+2}\right) + \cdots + \left(\frac{1}{b+1} - \frac{1}{2b+1}\right) + \left(\frac{1}{b+2} - \frac{1}{2b+2}\right) + \cdots\right.$$
$$\left.\left.\cdots + \left(\frac{1}{N-2b} - \frac{1}{N-b}\right) + \left(\frac{1}{N-2b+1} - \frac{1}{N-b+1}\right) + \cdots + \left(\frac{1}{N-b} - \frac{1}{N}\right)\right]\right\}.$$

After summing up the counts before the limit, cancellation all of the inner terms beyond the limit starting from $\frac{1}{b+1}$ up to $\frac{1}{N-b}$ and considering the facts that $\lim_{N\to\infty} \frac{1}{\ell} = \frac{1}{\ell}$ and $\lim_{N\to\infty} \frac{1}{N+\ell} = 0$ for any positive integer ℓ we get

$$\sum_{\substack{k=1\\k\neq b}}^{\infty}\frac{1}{k(k-b)}=\frac{-2}{b}\left(\frac{1}{b-1}+\frac{1}{b-2}+\cdots+\frac{1}{2}+1\right)+$$
$$+\frac{1}{b}\lim_{N\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{b}-\frac{1}{N-b+1}-\frac{1}{N-b+2}-\cdots-\frac{1}{N}\right)=$$
$$=\frac{-2}{b}\left(1+\frac{1}{2}+\cdots+\frac{1}{b-2}+\frac{1}{b-1}\right)+\frac{1}{b}\left(1+\frac{1}{2}+\cdots+\frac{1}{b}\right)=\frac{H_b-2H_{b-1}}{b}=$$
$$=\frac{-1}{b}\left(1+\frac{1}{2}+\cdots+\frac{1}{b-2}+\frac{1}{b-1}-\frac{1}{b}\right)=\frac{1-bH_{b-1}}{b^2}.$$

4 Numerical verification

We solve two problems – to determine the values of the sums

$$s(a,0)=\sum_{k=1}^{\infty}\frac{1}{k(k-a)}\qquad\text{and}\qquad s(0,b)=\sum_{\substack{k=1\\k\neq b}}^{\infty}\frac{1}{k(k-b)}$$

for negative integer a and positive integer b. For the numerical verification of the derived formulas (6) and (7) we choose these values: $a=-1,-2,-3,-10,-100$ and $b=1,2,3,10,100$. We use on the one hand an approximative direct evaluation of the sums

$$s(a,0,t)=\sum_{k=1}^{t}\frac{1}{k(k-a)}\qquad\text{and}\qquad s(0,b,t)=\sum_{\substack{k=1\\k\neq b}}^{t}\frac{1}{k(k-b)},$$

where $t=10^8$, using the computer algebra system Maple 16, and on the other hand the formula (6) for evaluation the sum $s(a,0)$ and the formula (7) for evaluation the sum $s(0,b)$.

For direct evaluation of the sums $s(a,0,10^8)$ and $s(0,b,10^8)$ we use the Maple commands illustrated by the following two examples, where $a=-10$ and $b=10$:

```
> evalf[8](sum(1/(k*(k+10)),k=1..10^8));
                                0.29289682
> evalf[8](harmonic(10)/10);
                                0.29289683
> evalf[8](sum(1/(k*(k-10)),k=1..9)+sum(1/(k*(k-10)),k=11..10^8));
                               -0.27289684
> evalf[8]((1-10*harmonic(9))/100);
                               -0.27289683
```

The approximative values of the sums $s(a,0,10^8)$ and $s(a,0)$ for $a=-1,-2,-3,-10,-100$ and the sums $s(0,b,10^8)$ and $s(0,b)$ for $b=1,2,3,10,100$, rounded to 8 decimals and obtained by the calculations made in the CAS Maple are written into the following Table 2, where the absolute errors of the calculations are all approximately equal to $1\cdot10^{-8}$:

a	-1	-2	-3	-10	-100
$s(a,0,10^8)$	0.99999999	0.74999999	0.61111110	0.29289682	0.05187377
$s(a,0)$	1.00000000	0.75000000	0.61111111	0.29289683	0.05187378
b	1	2	3	10	100
$s(0,b,10^8)$	0.99999999	-0.25000001	-0.38888890	-0.27289684	-0.05167379
$s(0,b)$	1.00000000	-0.25000000	-0.38888889	-0.27289683	-0.05167378

Tab. 2. Approximate values of the sums $s(a,0,10^8)$, $s(a,0)$, $s(0,b,10^8)$, $s(0,b)$.

5 Examples

Now, let us give some examples illustrating using formulas (1) to (7).

$$s(3,5)=\sum_{\substack{k=1\\k\neq 3,5}}^{\infty}\frac{1}{(k-3)(k-5)}=\frac{1}{5-3}(H_{3-1}-H_{5-1}+2H_{5-3}-2H_{5-3-1})=\frac{53}{24}\doteq 2.208,\text{ by (1).}$$

$$s(-5,-3)=\sum_{k=1}^{\infty}\frac{1}{(k+5)(k+3)}=\frac{1}{-3-(-5)}\big(H_{-(-5)}-H_{-(-3)}\big)=\frac{9}{40}\doteq 0.225,\text{ by (2).}$$

$$s(-5,-5)=\sum_{k=1}^{\infty}\frac{1}{(k+5)^2}=\frac{\pi^2}{6}-H_{-(-5),2}\doteq 1.645-\frac{5269}{3600}\doteq 0.181,\text{ by (3).}$$

$$s(5,5)=\sum_{\substack{k=1\\k\neq 5}}^{\infty}\frac{1}{(k-5)^2}=\frac{\pi}{2}+H_{5-1,2}\doteq 1.571+\frac{205}{144}\doteq 2.994,\text{ by (4).}$$

$$s(-3,5)=\sum_{\substack{k=1\\k\neq 5}}^{\infty}\frac{1}{(k+3)(k-5)}=\frac{\big(5+3\big)\big(H_{-(-3)}-H_{5-1}\big)}{\big(5+3\big)^2}=\frac{8\big(\frac{11}{6}-\frac{25}{12}\big)}{64}=\frac{-1}{32}\doteq -0.031,\text{ by (5).}$$

$$s(-5,0)=\sum_{k=1}^{\infty}\frac{1}{k)(k+5)}=\frac{H_{-(-5)}}{-(-5)}=\frac{\frac{137}{60}}{5}=\frac{137}{300}\doteq 0.457,\text{ by (6).}$$

$$s(0,5)=\sum_{\substack{k=1\\k\neq 5}}^{\infty}\frac{1}{k(k-5)}=\frac{1-5H_{5-1}}{5^2}=\frac{1-5\cdot\frac{25}{12}}{25}=\frac{-113}{300}\doteq -0.377,\text{ by (7).}$$

6 Conclusion

We deal with the sum $s(a,b)$ of the series of reciprocals of the quadratic polynomials with two integer roots a, b, i.e. with the series

$$\sum_{k=1}^{\infty}\frac{1}{(k-a)(k-b)},$$

where the index of summation $k\neq a$, if $a>0$, or $k\neq b$, if $b>0$.

This contribution pick up the threads of author's previous results stated in the papers [1], [2], [3], [4], [5] concerning non-zero integer roots and complete them with the case of zero roots. At first we remember the Euler's result

$$s(0,0)=\sum_{k=1}^{\infty}\frac{1}{(k-0)^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

and then derived the formulas for the sums $s(a,0)$, $a<0$, and $s(0,b)$, $b>0$:

$$s(a,0)=\sum_{k=1}^{\infty}\frac{1}{k(k-a)}=\frac{H_{-a}}{-a}\qquad\text{and}\qquad s(0,b)=\sum_{\substack{k=1\\k\neq b}}^{\infty}\frac{1}{k(k-b)}=\frac{1-bH_{b-1}}{b^2},$$

where H_n is the nth harmonic number. We verified these results by computing 10 sums (for $a=-1,-2,-3,-10,-100$ and $b=1,2,3,10,100$) by using the CAS Maple 16. We also stated that the last formula above can be written in the form

$$s(0,b)=\frac{H_b-2H_{b-1}}{b}.$$

The series of reciprocals of the quadratic polynomials with integer roots so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a simple formula.

References

[1] POTŮČEK, R.: *The sum of the series of reciprocals of the quadratic polynomial with different positive integer roots*. In: Matematika, informační technologie a aplikované vědy (MITAV 2016). Univerzita obrany, Brno, 2016, p. 32-43. ISBN 978-80-7231-464-5.

[2] POTŮČEK, R.: *The sum of the series of reciprocals of the quadratic polynomial with different negative integer roots*. In: Ratio Mathematica - Journal of Foundations and Applications of Mathematics, 2016, vol. 30, no. 1/2016, p. 59-66. ISSN 1592-7415.

[3] POTŮČEK, R.: *The sum of the series of reciprocals of the quadratic polynomials with double non-positive integer root*. In: Proceedings of the 15th Conference on Applied Mathematics APLIMAT 2016. Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, 2016, p. 919-925. ISBN 978-80-227-4531-4.

[4] POTŮČEK, R.: *The sum of the series of reciprocals of the quadratic polynomials with double positive integer root*. In: MERAA – Mathematics in Education, Research and Applications, 2016, vol. 2, no. 1. Slovak University of Agriculture in Nitra, 2016, p. 15-21. ISSN 2453-6881.

[5] POTŮČEK, R.: *The sum of the series of reciprocals of the quadratic polynomials with one negative and one positive integer root*. In: Proceedings of the 16th Conference on Applied Mathematics APLIMAT 2017. Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, 2017, p. 1252-1253. ISBN 978-80-227-4650-2.

[6] WIKIPEDIA CONTRIBUTORS: *Harmonic number*. Wikipedia, The Free Encyclopedia, [online], [cit. 2017-11-30]. URL: https://en.wikipedia.org/wiki/Harmonic_number.

Current address

Potůček Radovan, RNDr. Ph.D.
Department of Mathematics and Physics
Faculty of Military Technology, University of Defence in Brno
Kounicova 65, 662 10 Brno, Czech Republic
Tel. number: + 420 973 443 056, e-mail: Radovan.Potucek@unob.cz