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WEAK PROPERTY B AND PROPERTY B OF SYSTEM OF FOUR NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We investigate a system of the four nonlinear difference equations, where the first equation obtain a neutral term. We state sufficient conditions for system to have the weak property B and property B.

Keywords: Property B, nonoscillatory solutions, oscillatory solution, difference equation

Mathematics subject classification: Primary 39A10; Secondary 39A21.

1 Introduction

In this paper, we study asymptotic behavior of solutions of a four-dimensional system

$$\begin{aligned} \Delta\left(x_n + p_n x_{n-\sigma}\right) &= A_n\, f_1\left(y_n\right) \\ \Delta y_n &= B_n\, f_2\left(z_n\right) \\ \Delta z_n &= A_n\, f_3\left(w_n\right) \\ \Delta w_n &= D_n\, f_4\left(x_{\gamma_n}\right), \end{aligned} \tag{S}$$

where $n \in \mathbb{N}_0 = \{n_0, n_0+1, ...\}$, n_0 is a positive integer, σ is a nonnegative integer, $\{A_n\}, \{B_n\}, \{D_n\}$ are positive real sequences defined for $n \in \mathbb{N}_0$. Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$.

The sequence $\gamma : \mathbb{N} \to \mathbb{N}$ satisfies

$$\gamma_n \geq n + \sigma. \tag{H1}$$

The most common form of this sequence is $\gamma_n = n \pm \tau$, where $\tau \in \mathbb{N}$.

The sequence $\{p_n\}$ is a sequence of the real numbers and it satisfies

$$\lim_{n\to\infty} p_n = P, \text{ where } |P| < 1. \tag{H2}$$

Functions $f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, .., 4$ satisfy

$$\frac{f_i(u)}{u} \geq 1,\ u \in \mathbb{R}\backslash 0. \tag{H3}$$

Nonlinear difference systems or difference equations are often studied when

$$\sum_{n=n_0}^{\infty} A_n = \infty, \quad \sum_{n=n_0}^{\infty} B_n = \infty, \tag{H4}$$

which is called that the system (S) is in the canonical form. In this paper, we study the system under the following conditions

$$\sum_{n=n_0}^{\infty} A_n = \infty, \quad \sum_{n=n_0}^{\infty} B_n < \infty, \quad \sum_{n=n_0}^{\infty} B_n \left(\sum_{i=n_0}^{n-1} A_i \right) = \infty. \tag{H5}$$

By a solution of the system (S) we mean a vector sequence (x, y, z, w) which satisfies the system (S) for $n \in \mathbb{N}_0$. We investigate oscillatory or nonoscillatory solutions.

Definition 1. *The component x is said to be* ***nonoscillatory*** *if there exists $n_1 \geq n_0$ such that $x_n \geq 0$ (respectively $x_n \leq 0$) for all $n \geq n_1$. A solution of (S) is said to be nonoscillatory if all of its components x, y, z, w are nonoscillatory.*

Definition 2. *The component x is said to be* ***oscillatory*** *if for any $n_1 \geq n_0$ there exists $n \geq n_1$ such that $x_{n+1}x_n < 0$. A solution of (S) is said to be oscillatory if all of its components x, y, z, w are oscillatory.*

We study when (S) has a Property B or Weak property B. Property B is defined in accordance with those for the higher-order differential equations or for the system of differential equations, see [9] and references therein.

Definition 3. *The system (S) has* ***weak property B*** *if every nonoscillatory solution of (S) satisfies*

$$x_n z_n > 0 \quad \textit{and } y_n w_n > 0 \quad \textit{for large } n. \tag{1}$$

Definition 4. *The system (S) has* ***property B*** *if any of its solutions either is oscillatory or satisfies either*

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = \infty, \tag{2}$$

or

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0. \tag{3}$$

Solutions satisfying (1) and $x_n y_n > 0$ are called *strongly monotone solutions*, while solutions satisfying (1) and $x_n y_n < 0$ are called *Kneser solutions*.

The system (S) can be easily rewritten as a fourth-order nonlinear neutral difference equation. Equations with quasi-differences have been widely studied in the literature; see, for example, [1]–[7], [10]. Equations of this form appear in the discretization process for solving models concerning physical, biological, and chemical henomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement.

The aim of this paper is to extend our results about asymptotic behavior of nonoscillatory solutions of (S). We try to extend our results from [8]. We give sufficient conditions that (S) has weak property B and property B for the system (S) with the different assumptions then in previous. In [8], we establish sufficient conditions for the system (S) to have weak property B and we suppose that (S) is in the canonical form with positive sequence $\{p_n\}$. In this paper, we have the symetric operator of (S) and we modify the conditions (H4) into (H5) and sequence γ_n in order to have more general conditions.

2 Types of nonoscillatory solutions

If the system (S) has a solution (x, y, z, w), then it has the solution $(-x, -y, -z, -w)$ as well. Thus, throughout the paper, we can focus on solutions whose first component is eventually positive for large n.

We use the notation

$$s_n = x_n + p_n x_{n-\sigma}, \tag{4}$$

where $n \in \mathbb{N}_0$.

In proofs of our theorems we use the following lemma, which was proved in [8] for p positive, but it is easy to see that it holds for (H2) as well.

Lemma 1. *[8, Lemma 1] Let $\{x_n\}$ be eventually positive sequence and $\{p_n\}$ satisfies* (H2)*, $n \in \mathbb{N}_0$. Let $\{s_n\}$ be the sequence defined by* (4)*. Then $\{x_n\}$ is bounded if and only if $\{s_n\}$ is bounded. Moreover, if $\{s_n\}$ is positive and increasing for large n, then*

$$x_n \ge s_{n-\sigma}(1 - p_n) \quad \textit{for large } n. \tag{5}$$

The following lemma describes the possible types of nonoscillatory solutions.

Lemma 2. *Assume* (H4)*. Then any nonoscillatory solution (x, y, z, w) of (S) with eventually positive x is one of the following types:*

type (a) $x_n > 0$ $y_n > 0$ $z_n > 0$ $w_n > 0$ *for large n,*
type (b) $x_n > 0$ $y_n > 0$ $z_n > 0$ $w_n < 0$ *for large n,*
type (c) $x_n > 0$ $y_n < 0$ $z_n > 0$ $w_n < 0$ *for large n,*
type (d) $x_n > 0$ $y_n < 0$ $z_n < 0$ $w_n < 0$ *for large n,*
type (e) $x_n > 0$ $y_n > 0$ $z_n < 0$ $w_n < 0$ *for large n.*

Proof. Let (x, y, z, w) be a nonoscillatory solution of (S) such that $x_n > 0$ for large n. There are eight possible types of these solutions. We prove that solutions of the following types do not exist.

type (i) $x_n > 0$ $y_n > 0$ $z_n < 0$ $w_n > 0$ for large n,
type (ii) $x_n > 0$ $y_n < 0$ $z_n < 0$ $w_n > 0$ for large n.
type (iii) $x_n > 0$ $y_n < 0$ $z_n > 0$ $w_n > 0$ for large n.

Assume that there exist $n_1 \in \mathbb{N}_0$ and a solution such that $z_n < 0$, $w_n > 0$ for $n \ge n_1 \ge n_0$. From the fourth equation of (S) we have $\Delta w_n > 0$ and this implies that there exists $k > 0$ such that $w_n \ge k$ for large n. Using (H3) we have $f_3(w_n) \ge w_n \ge k$. By the summation of the third equation of (S) we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i f_3(w_i) \ge k \sum_{i=n_0}^{n-1} A_i.$$

Passing $n \to \infty$, we get a contradiction with the fact that $z_n < 0$. This excludes solutions of types (i) and (ii).

Assume that (x, y, z, w) is a type (iii) solution. Since w is positive and increasing, there exists $k > 0$ such that $w_n \ge k$ for large n. By the summation of the third equation of (S) we get

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i f_3(w_i) \ge \sum_{i=n_0}^{n-1} A_i w_i \ge k \sum_{i=n_0}^{n-1} A_i.$$

Using the summation of the second equation of (S) we have

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} B_i f_2(z_i) \geq \sum_{i=n_0}^{n-1} B_i z_i \geq k \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} A_j \right).$$

Passing $n \to \infty$ we get the contradiction with the negativity of y.

□

By Definition 3, the system (S) has the weak property B if there exist only nonoscillatory solutions of type (a) and (c). Solutions of type (a) are called strongly monotone and solutions of type (c) are called Kneser solutions.

Lemma 3. *Any solution of type (a) satisfies*

$$\lim_{n\to\infty} x_n = \infty, \ \lim_{n\to\infty} z_n = \infty. \tag{6}$$

Proof. Let (x, y, z, w) be a solution of type (a). Since y is positive and increasing, there exists $k > 0$ such that $y_n \geq k$ for large n. By the summation of the first equation of (S) we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1(y_i) \geq \sum_{i=n_0}^{n-1} A_i y_i \geq k \sum_{i=n_0}^{n-1} A_i. \tag{7}$$

Passing $n \to \infty$ we get $s_n \to \infty$. Lemma 1 implies that s is unbounded if and only if x is unbounded. Therefore $\lim_{n\to\infty} x_n = \infty$.

Since w is eventually positive increasing, thus there exists $l > 0$ such that $w_n \geq l$ for large n. By the summation of the third equation of (S) we obtain that $z_n \to \infty$ for $n \to \infty$.

□

Lemma 4. *Any solution of type (c) satisfies*

$$\lim_{n\to\infty} w_n = 0. \tag{8}$$

Proof. Assume that the solution (x, y, z, w) is of type (c). Since w is eventually negative increasing, there exists $\lim_{n\to\infty} w_n = l \leq 0$. Suppose $l < 0$. By the summation of the third equation of (S) we obtain a contradiction with the positivity of z. Therefore $\lim_{n\to\infty} w_n = 0$. □

3 Weak property B and property B

The first theorem gives the simple criterion that the system (S) has property B, the proof of the theorem is omitted, we can proceed the similar way as in [8].

Theorem 1. *Assume* (H5). *If*

$$\sum_{n=n_0}^{\infty} D_n = \infty \tag{9}$$

holds, then the system (S) has property B.

In view of Theorem 1, in the sequel, we assume $\sum_{n=n_0}^{\infty} D_n < \infty$.

Theorem 2. *Let* (H1) - (H3) *hold. If*

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{i-1} A_j \right) = \infty, \tag{10}$$

then the system (S) has weak property B.

Proof. The weak property B means that there are no solutions of type (b), (d) and (e). Therefore we have to exclude these solutions. Assume that (x, y, z, w) is a type (b) solution. By (7) we get $\lim_{n\to\infty} x_n = \infty$, thus there exists $k > 0$ such that $x_n \geq k$ for large n. By the summation of the fourth equation of (S) we get

$$w_\infty - w_n = \sum_{i=n}^{\infty} D_i f_4 (x_{\gamma_i}) \geq \sum_{i=n}^{\infty} D_i x_{\gamma_i} \geq k \sum_{i=n}^{\infty} D_i,$$

Using the summation of the third equation of (S) we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i f_3 (w_i) \leq \sum_{i=n_0}^{n-1} A_i w_i,$$

$$-z_n + z_{n_0} \geq \sum_{i=n_0}^{n-1} A_i (-w_i) \geq k \sum_{i=n_0}^{n-1} A_i \left(\sum_{j=i}^{\infty} D_j \right).$$

Passing $n \to \infty$ and using the change of summation

$$\sum_{i=n_0}^{\infty} A_i \left(\sum_{j=i}^{\infty} D_j \right) = \sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{i-1} A_j \right) = \infty,$$

we get the contradiction with the boundedness of z. Thus, solutions of type (b) do not exist.

Assume that (x, y, z, w) is a type (d) solution. Since y is negative and decreasing, there exists $k < 0$ such that $y_n \leq k$ for large n. By the summation of the first equation of (S) we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1 (y_i) \leq \sum_{i=n_0}^{n-1} A_i y_i \leq k \sum_{i=n_0}^{n-1} A_i. \tag{11}$$

Using the summation of the fourth equation of (S) we have

$$w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i f_4 (x_{\gamma_i}) \geq \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \tag{12}$$

From (H2) and (4) we have $s_n \geq x_n - x_{n-\sigma} \geq -x_{n-\sigma}$ for large n. Thus

$$x_n \geq -s_{n+\sigma}. \tag{13}$$

Using this and (12) and (11) we get

$$w_n - w_{n_0} \geq \sum_{i=n_0}^{n-1} D_i (-s_{\gamma_i+\sigma}) \geq -k \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i+\sigma-1} A_j \right).$$

Passing $n \to \infty$ we get from (H1) that (10) implies

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{\gamma_i+\sigma-1} A_j \right) = \infty.$$

Thus, we get the contradiction with the negativity of w.

Assume that (x, y, z, w) is a type (e) solution. Since z is negative and decreasing, there exists $h < 0$ such that $z_n \le h$ for large n. By the summation of the second equation of (S) we get

$$y_\infty - y_n = \sum_{i=n}^{\infty} B_i f_2 (z_i) \le \sum_{i=n}^{\infty} B_i z_i \le h \sum_{i=n}^{\infty} B_i.$$

Using the summation of the first equation of (S) we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1 (y_i) \ge \sum_{i=n_0}^{n-1} A_i y_i \ge -h \sum_{i=n_0}^{n-1} A_i \left(\sum_{j=i}^{\infty} B_j \right). \tag{14}$$

In case $p_n \le 0$, we get $s_n \le x_n$ from (4). Using this fact, the summation of the fourth equation of (S) and the estimation (14) we obtain

$$w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i f_4 (x_{\gamma_i}) \ge \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \ge \sum_{i=n_0}^{n-1} D_i s_{\gamma_i},$$

$$w_n - w_{n_0} \ge -h \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i-1} A_j \left(\sum_{k=j}^{\infty} B_k \right) \right). \tag{15}$$

In case $p_n > 0$ we use (5) and we get

$$w_n - w_{n_0} \ge \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \ge \sum_{i=n_0}^{n-1} D_i s_{\gamma_i-\sigma}(1 - p_{\gamma_i}),$$

$$w_n - w_{n_0} \ge -h(1-P) \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i-\sigma-1} A_j \left(\sum_{k=j}^{\infty} B_k \right) \right). \tag{16}$$

Passing $n \to \infty$ we get the contradiction with the negativity of w in both cases (15), (16). □

Theorem 3. *Let* (10) *hold. In addition, if*

$$\sum_{i=n_0}^{\infty} A_i \left(\sum_{j=i}^{\infty} D_j \right) = \infty \tag{17}$$

holds, then the system (S) has property B.

Proof. By Theorem 2, the system (S) has only solutions of type (a) and (c). First, assume that (x, y, z, w) is a solution of type (a). Thus, w is positive increasing and there exists a constant $k_1 > 0$ such that $w_n \geq k_1$ for large n. From the third equation of (S) we get

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i f_3(w_i) \geq \sum_{i=n_0}^{n-1} A_i w_i \geq k_1 \sum_{i=n_0}^{n-1} A_i.$$

Substituting this into the second equation of (S) we obtain

$$y_n - y_{n_0} = \sum_{i=n_0}^{n-1} B_i f_2(z_i) \geq \sum_{i=n_0}^{n-1} B_i z_i \geq k_1 \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} A_j \right).$$

Passing $n \to \infty$ we have $y_n \to \infty$.

Taking into account $\lim(1 - p_n) = 1 - P > 0$, there exists $p > 0$ such that $1 - p_n \geq p$, for large n. Using the summation of the fourth equation of (S) and (5) we have

$$w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i f_4 \left(x_{\gamma_i}\right) \geq \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \geq \sum_{i=n_0}^{n-1} D_i s_{\gamma_i - \sigma}(1 - p_{\gamma_i}) \geq \tag{18}$$

$$\geq p \sum_{i=n_0}^{n-1} D_i s_{\gamma_i - \sigma} \geq kp \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i - \sigma - 1} A_j \right).$$

From (18) we get $w_n \to \infty$ passing $n \to \infty$.

Now, assume that (x, y, z, w) is a solution of type (c). Assume that $\lim x_n = t_1 \geq 0$. First, assume that $t_1 > 0$. Using the summation of the third and the fourth equation of (S) we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} A_i f_3(w_i) \leq -t_1 \sum_{i=n_0}^{n-1} A_i \left(\sum_{j=i}^{\infty} D_j \right).$$

Passing $n \to \infty$ we get the contradiction with the boundedness of z, therefore $\lim_{n\to\infty} x_n = 0$.

Now, assume that $\lim y_n = t_2 \leq 0$. First, assume that $t_2 < 0$. Using the summation of the first and the fourth equation of (S) we obtain

$$w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i f_4(x_{\gamma_i}) \geq \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \geq \sum_{i=n_0}^{n-1} D_i (-s_{\gamma_i + \sigma}) \geq -t_2 \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i + \sigma - 1} A_j \right).$$

Passing $n \to \infty$ we get the contradiction with the boundedness of w, therefore $\lim_{n\to\infty} y_n = 0$.

Finally, assume that $\lim z_n = t_3 \geq 0$. First, assume that $t_3 > 0$. Using the summation of the first and the second equation of (S) we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1(y_i) \leq -t_3 \sum_{i=n_0}^{n-1} A_i \left(\sum_{j=i}^{\infty} B_j \right).$$

Since

$$\sum_{i=n_0}^{\infty} A_i \left(\sum_{j=i+1}^{\infty} B_j \right) = \sum_{i=n_0}^{\infty} B_i \left(\sum_{j=n_0}^{n-1} A_j \right) = \infty,$$

then we get the contradiction with the boundedness of s. Since x is bounded, then s is bounded as well. Therefore $\lim_{n\to\infty} z_n = 0$.

Now, we get the assertion by Lemma 3, Lemma 4 and Definition 4.

□

4 Conclusion

We found conditions for the system (S) to have proberty B. We extended the conditions from [8]. Now, we can investigate the system with the different conditions or it could be interesting to find the sufficient conditions for system (S) to have any type of the solutions.

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