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USE OF STRUCTURAL DYNAMIC MODIFICATION TO REDUCING VIBRATION LEVEL

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Abstract. Dynamic parameters tuning is a required operation when significant vibration of a construction component is expected. Structural dynamic modification represents modal-spectral parameters changes by adding substructures to the original to avoid resonances and reduce the level of vibration. Those substructures, which influent mass, stiffness and damping matrices of the original system, can increase or decrease natural frequencies and influents corresponding mode shapes according to the desire behaviour of the structure. Additional tuning of dynamic parameters performed by structural dynamic modification is a useful operation to adapt the structure to changed operation conditions. Modal and spectral parameters of the modified structure can be determined by analytical approach using structural coefficient matrices of modifying substructures and modal parameters of the original structure, that can be obtained experimentally

Keywords: structural dynamic modification, modal synthesis, system DOF reducing, dynamic parameters tuning

Mathematics Subject Classification: Primary 15A18, 65E99

1 Introduction

A common occurrence in engineering practice are undesirable levels of vibration in the structures of machinery, which decrease their functionality, safety, reliability and service life. Current trends in the dynamic operation of machinery inherently generate such undesirable effects. That is to say, increasing the operational capacity of the machine (higher speeds, higher loads, more changes in operational regimes, etc...) are financially counterproductive to any desired savings in the material/technological realization of such structures. The dynamic properties of the machines structure itself, to an extent, affects the level of vibration of each of its individual parts. In the research stage, it is now necessary to extensively analyse/synthesize the dynamic properties of the machine and its structure followed by the optimization of significant parameters. Therefore an effective methodology to push (decrease) any high level

of vibration in critical parts of the machines structure is becoming increasingly more interesting to satisfy more demanding technological and economic design requirements.
In the design of a structure which must vibrate within acceptable levels during operation, an suitable (effective) concept must be chosen. For example, on the basis of numeric analysis and optimization of individual structural components, it is possible to create a real structure which satisfies the operational conditions set upon it. But in general, the real structure partially exhibits differing properties than those predicted computationally, either due to inappropriate simplification or inaccurate physical or geometrical parameters. It is therefore necessary to modify critical structural elements in order to fulfil the desired properties. To do this it is possible to use some correction method for the mathematical models. With them it is possible to refine the parameters of critical nodes in the structure to reflect the operation of the real structure. Afterwards it is necessary to repeat the optimization procedure of the revised structural parameters such that, based on them, the built structure reflects the required properties.

Another possible approach to modify the dynamic properties of the structure is through modal synthesis. This approach combines the modal properties of the real structure obtained through measurements and the modal properties of additional components obtained computationally or through measurements. This approach is particularly effective if the computational model of the built structure is incorrect. Through optimization of the additional components it is possible to obtain the desired properties of a modified structure while reducing the computational requirements and increasing accuracy of the results. The aforementioned approach is valid for a variety of applications, most notably the automotive and aerospace industries amongst many others.

2 Modelling a structure with a vibroinsulating layer made of aluminium foam

To reduce unacceptable levels of vibration, layers of different material are usually used to reduce the levels of vibration. Materials continue to evolve. New materials have not only good damping properties, but their behaviour also show other favourable characteristics. Such materials are typically difficult and expensive to produce, however the economic benefits from improving the properties if a structure far outweigh the initial cost for their production [1], [3].

2.1 Properties of aluminium foam

Metal and aluminium foams are amongst one group of materials with special properties. Their combination of beneficial physical properties allows them to be useful for a wide variety of applications in practice. The main benefit is in their low weight, high stiffness, and good damping properties. They are materials with many beneficial properties and wide spectrum of application. Even though these materials are relatively expensive the appropriate design of geometrical parameters and applying them in suitable locations, can effectively ensure the required stiffness and damping parameters to be achieved with a minimal use of material. A large part of practical calculations lays in the determination of frequency transfer (modal properties). Determining the modal properties for a specific position and geometrical

parameter of the vibro-isolating layers subsequently followed by the processing of the results, yields optimal parameters.
As was discussed, aluminium foams are used also as a material with properties of vibro-isolation. Tab. 1 shows a comparison of material properties between conventional materials and aluminium foam (Alulight from Alulight International GmbH) [1], [18], [4]. In respect the properties of Alulight, it is possible to apply where relatively stiff structural components are needed with a minimum of added weight, or where heat resistant, high stiffness, low density structures are required.

Material	Density ρ [kg/m^2]	Modulus E [GPa]	E 10^{-5} [GPa.kg^2/m^6]
Alulight	500	5	2,0
Epoxy	1300	3	0,3
steel	7800	210	0,4
aluminium	2700	69	1,0
glass	2500	70	1,1
concrete	2500	50	0,8

Tab 1. Comparison of physical properties of aluminium foam and conventional materials.

2.2 Modelling of layered structures

Currently, the finite element method is used when modelling vibration of a machine structures [4], [5], [19], [13]. Using this method, it is possible to solve for layered materials and directly use elements respecting the required inertial, stiffness and damping properties. In the case of a thin, layered beam and considering the bending vibration, the element matrices are written in the form:

$$\mathbf{M}_{\mathrm{E}} = m_{\mathrm{E}} \begin{bmatrix} \frac{13}{35} & & & \\ \frac{11l}{420} & \frac{l^2}{105} & sym & \\ \frac{9}{700} & \frac{13l}{420} & \frac{13}{35} & \\ \frac{-13l}{420} & \frac{-l^2}{140} & \frac{-11l}{210} & \frac{l^2}{105} \end{bmatrix}$$

$$\mathbf{K}_{\mathrm{E}} = \frac{2EJ}{l^3} \begin{bmatrix} 6 & & & \\ 2l & 2l^2 & sym & \\ -6 & -2l & 6 & \\ 2l & l^2 & -2l & 2l^2 \end{bmatrix} \tag{1}$$

where the displacement and rotation within nodes of the element are represented by the vector: $[y_j, \varphi_k, y_j, \varphi_k,]^T$, where m_E is the mass, E is the equivalent module of elasticity, J is the equivalent moment of inertia of the cross-section area and l is the length of the layered element (Fig. 1). The equivalent modulus of elasticity is the beams module of elasticity $E = E_1$. The moment of inertia of the cross-sectional area is the moment of inertia of the transformed cross-sectional area of the added layer [13], [6].

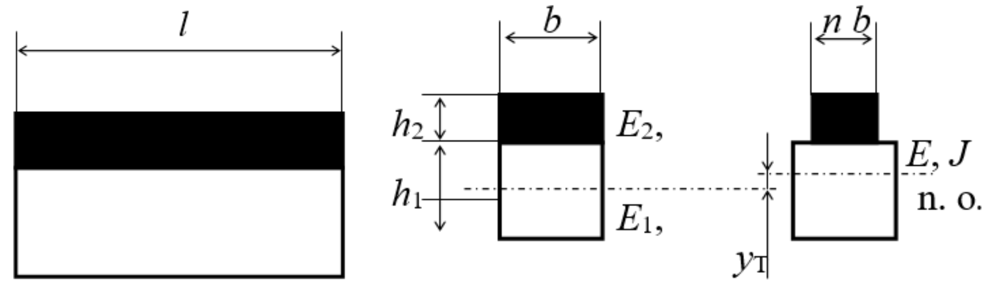


Fig. 1. Layered beam element with cross section and transformed cross section.

The element damping matrix is expressed in proportional terms $\mathbf{B}_E = \beta \mathbf{K}_E$, where $\beta = \eta/\omega$, η is the loss coefficient, and ω is the angular excitation frequency. The loss coefficient is determined experimentally and is given by the manufacturer. Tab. 2 shows the necessary material properties of aluminium foam to calculate the element matrix [1], [18], [2].

Density ρ	kg/m^3	500	750	1000
Module of Elasticity E	GPa	5	9	14
Loss coefficient η	-	0,003	0,004	0,004

Tab 2. Comparison of physical properties of aluminium.

The complete FEM beam model is created by connecting elements in corresponding node points (see Fig. 2).

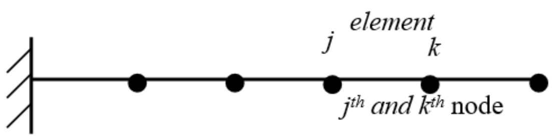


Fig. 2. FE model of the beam.

Its mathematical model has the form:

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{B}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{f} \tag{2}$$

where matrices for mass **M**, stiffness **K** and damping **B** have a striped character and make up an overlapping element matrix in the rows and columns where elements have a common displacement and rotation (the same node). Simultaneously in the the element matrices the first two rows are tied to the displacement and rotation in the j^{th} node and the rest are tied to the displacement and rotation of the k^{th} node. The same applies to the columns of the element matrices.

3 Reducing the Systems Degrees of Freedom

FEM models that closely represent the dynamic behaviour within the given frequency range tend to have many degrees of freedom (DOF) and therefore the coefficient matrices of the system are large. However, when measuring, it is necessary to analyse the measurement points of the structure whose DOF (for example rotation) are not measurable or poorly measured. Therefore, it is beneficial to reduce the model as to be consistent with the experiment. This means that the computational model will be represented by movements which are measured and are true to the tested results within the given frequency range. There are many methods which can be used to reduce such a model [7], [9] – [12], [16].
Methods for reduction, in principle, are distinguished by the approximation of insignificant common motions in the system ("insignificant – omitted - secondary" DOF) within the considered range of frequencies, and the significant common dynamic motion ("dominant - primary - main" DOF) determined based on physical/modal properties of the system (Fig. 3). In this way the insignificant (secondary) DOF are omitted based on the approximation function chosen based on identifying the (primary) DOF.

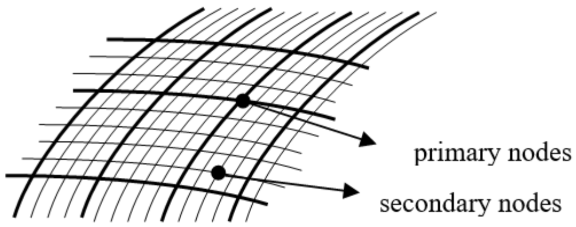


Fig. 2. FE model with primary and secondary nodes.

In principle, the shown method is based on the election of the transformation matrix **T** of type *n*, *r* and ranking *m*, for the transformation of coordinates $\mathbf{q}(t)$ of the model by dimension *n* to the coordinate $\mathbf{q}_R(t)$ of the model with dimension *r*, where:

$$\mathbf{q}(t) = \mathbf{T}\,\mathbf{q}_R(t). \tag{3}$$

Such a transformation matrix characterizes the relationship between secondary and primary coordinates of the system. Modifying relation (2) using this transformation will yield the reduced model in the form:

$$\mathbf{M}_R\,\ddot{\mathbf{q}}_R(t) + \mathbf{B}_R\,\dot{\mathbf{q}}_R(t) + \mathbf{K}_R\,\mathbf{q}_R(t) = \mathbf{f}_R\,(t), \tag{4}$$

where

$$\mathbf{M}_R = \mathbf{T}^T\,\mathbf{M}\,\mathbf{T}\,;\ \mathbf{B}_R = \mathbf{T}^T\,\mathbf{B}\,\mathbf{T}\,;\ \mathbf{K}_R = \mathbf{T}^T\,\mathbf{K}\,\mathbf{T}\,;\ \mathbf{f} = \mathbf{T}^T\,\mathbf{f}(t). \tag{5}$$

The accuracy of any method depends on the choice of a suitable transformation matrix **T**, which should ensure that the reduced model to the degree r remains modally and spectrally (in r selected coordinates, $m \leq r$ selected Eigen modes) true to the original model to the n^{th} degree.
It follows from the modal analysis of such a system that the reduction will be more accurate depending on how correctly the transformation equation between elements of the structures Eigen modes $\mathbf{v}_j$ (n^{th} amount) and from them the selected components $\mathbf{v}_{Rj}$ (r amount) [9], [11]. Afterward, if:

$$\mathbf{v}_j = \mathbf{T}\,\mathbf{v}_{\mathrm{R}j}, \tag{6}$$

then both of the following relations apply:

$$(\mathbf{K} + s_j\,\mathbf{B} + s_j^2\,\mathbf{M})\,\mathbf{v}_j = \mathbf{0},\ (\mathbf{K}_\mathrm{R} + s_j\,\mathbf{B}_\mathrm{R} + s_j^2\,\mathbf{M}_\mathrm{R})\,\mathbf{v}_{\mathrm{R}j} = \mathbf{0}, \tag{7}$$

where s_j (in amount $m \leq r$) are the values from the considered frequency range.
In cases where, for example, the system with transformrmation matrix **T** (System Equivalent Reduction and Expansion Process – SEREP method) in the form:

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} \\ \mathbf{V}_{0\mathrm{O}}\,\mathbf{V}_{0\mathrm{R}}^{+} \end{bmatrix},\ \mathbf{V}_0 = \mathbf{T}\,\mathbf{V}_{0\mathrm{R}}, \tag{8}$$

where $\mathbf{V}_0 \in R^{n,m}$ is a matrix with m – significant Eigen modes , which are applicable for specific directional actions and excitation frequency ranges. Its submatrix $\mathbf{V}_{0\mathrm{R}} \in R^{r,m}$ is composed of elements characterized by r – most significant common deviations of the structure (usually $m \leq r << n$) and its sub matrix $\mathbf{V}_{0\mathrm{O}} \in R^{o,m}$ is composed of elements characterized by $o = n - r$ remaining elements of this set.
Using the transformation matrix **T,** the relationship for orthonormality are satisfied only in the considered frequency range. In the experimental verification of the aforementioned reduction method, if the sub matrix $\mathbf{V}_{0\mathrm{R}}$ is determined experimentally and sub matrix $\mathbf{V}_{0\mathrm{O}}$ is determined analytically, then the given relations are satisfied in respect to the minima of the quadratic standards of such a regression model. This problem, in general, occurs as a result of an inconsistent analytical model to the r - degree with the n^{th} degree of the model (that is with errors in modelling and measuring) [10, 8, 17, 14]. Such a transformation matrix must represent the real properties of the structure, which is used for example for the correction of the mathematical models of the systems based on measured data.

Inconsistencies often occur due to errors in the analytical modelling of the structure and in measurements during experimental tests. Therefore, in order to minimize computational errors in terms of problem verification, identification, system corrections (modal synthesis), it is necessary to choose the elements of the modal vector which dominantly represent the orthonormal properties of the system. This will guarantee the rank m of the transformation matrix **T**. Their selection can be objectivised by a singular or QR (Gramm-Schmidt orthogonalization process) decomposition of the modal matrix $\mathbf{V}_0$. Any sudden decrease (large difference) between j and k - th singular values of the modal matrix ($\sigma_1 > \sigma_2 \ldots \sigma_j >> \sigma_k\ \sigma_{k+1} > \ldots$) [7], [9], [12] will not occur in the case where the analytically obtained submatrices are

given because in this case the errors from measurements do not occur, however matrix $\mathbf{V}_0$ must be regular (its rank must be m).
Since the modal analysis is relatively taxing computationally, the transformation matrix **T** can be expressed by the already computed coefficient matrices of the system. The following is true for the system (2) (matrices assumed to be symmetric and **B=0** with homogeneous solution), expanded to block form:

$$\left(\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix} - \omega_{0j}^2 \begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}\right)\begin{bmatrix}{}^{r}\mathbf{v}_{0j}\\ {}^{o}\mathbf{v}_{0j}\end{bmatrix} = \mathbf{0}\ ,\ \mathbf{V}_0 = \{\mathbf{v}_{0j}\},\ \mathbf{v}_{0j} = \begin{bmatrix}{}^{r}\mathbf{v}_{0j}\\ {}^{o}\mathbf{v}_{0j}\end{bmatrix} \tag{9}$$

where $\mathbf{K}_{11}, \mathbf{M}_{11} \in R^{r,r}$, $\mathbf{K}_{12} = \mathbf{K}_{21}^{T}$, $\mathbf{M}_{12} = \mathbf{M}_{21}^{T} \in R^{r,o}$, $\mathbf{K}_{22}, \mathbf{M}_{22} \in R^{o,o}$, ${}^{r}\mathbf{v}_{0j} \in R^{r}$, ${}^{o}\mathbf{v}_{0j} \in R^{o}$.
then for the transformation matrix the following applies:

$${}^{o}\mathbf{v}_{0j} = \mathbf{T}_{D}\, {}^{r}\mathbf{v}_{0j} = -(\mathbf{K}_{22} - \omega_{0j}^2\, \mathbf{M}_{22})^{-1} (\mathbf{K}_{21} - \omega_{0j}^2\, \mathbf{M}_{21})\, {}^{r}\mathbf{v}_{0j},\ \mathbf{T} = \begin{bmatrix}\mathbf{I}\\ \mathbf{T}_{D}\end{bmatrix} \tag{10}$$

Relation (9) approximates the static condition – negligible inertial effects (Guayan reduction method) in the form [6]:

$${}^{o}\mathbf{v}_{0j} = -\mathbf{K}_{22}^{-1}\, \mathbf{K}_{21}\, {}^{r}\mathbf{v}_{0j}\ ,\ \mathbf{T} = \mathbf{T}_{G} = -\mathbf{K}_{22}^{-1}\, \mathbf{K}_{21}, \tag{11}$$

where the reduced matrix of the model is then:

$$\mathbf{K}_R = \mathbf{K}_{11} - \mathbf{K}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{K}_{21},\ \mathbf{B}_R = \mathbf{B}_{11} - \mathbf{K}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{B}_{21} - \mathbf{B}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{K}_{21} + \mathbf{K}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{B}_{22}\,\mathbf{K}_{22}^{-1}\,\mathbf{K}_{21}$$
$$\mathbf{M}_R = \mathbf{M}_{11} - \mathbf{K}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{M}_{21} - \mathbf{M}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{K}_{21} + \mathbf{K}_{12}\,\mathbf{K}_{22}^{-1}\,\mathbf{M}_{22}\,\mathbf{K}_{22}^{-1}\,\mathbf{K}_{21}, \tag{12}$$

This approximation becomes more accurate the lower the natural angular frequencies ω_{Red1} if

$$(\mathbf{K}_{22} - \omega_0^2\, \mathbf{M}_{22}) = \mathbf{0} \tag{13}$$

is greater than the maximum angular frequency of the considered excitation frequency range. It is therefore obvious that the selection of coordinates, which will be considered in further relations (primary), effect the accuracy of this method. Whereas the residual (secondary) coordinates are from the area of high stiffness and low mass. The stated selection can be objectively expressed in cases where the coefficient matrices have a dominant diagonal character. Any *j-th* coordinate which satisfies the following condition can be considered as residual (secondary) coordinates:

$$\frac{k_{jj}}{m_{jj}} > \omega_{max}. \tag{14}$$

The static condition (11) can be precisely satisfied in cases where, for example, the model is obtained by FEM with diagonal mass matrices $\mathbf{M}_{12} = \mathbf{M}_{21}^{T} = \mathbf{0}$, whereas the mass of the system is concentrated only at specific nodes, while the coordinates of the residuals are represented by the secondary coordinates connected with $\mathbf{M}_{22} = \mathbf{0}$.

In more complex systems, where it is necessary to reduce the dynamic properties of individual connecting between components a combination of this and the previously mentioned (Craig – Bampton method) can be used [7], [12] . In the system where the force effects act on the boundary of individual substructures $\mathbf{q}_R$ (in the primary coordinates) and the dynamic properties of the individual substructure $\mathbf{q}_O$ are represented by their modal properties (secondary coordinates), then for a single *j*-th substructure [10, 14] it then applies:

$$\mathbf{q}_{Oj} = \mathbf{V}_{0j}\, \tilde{\mathbf{q}}_{Oj} + \mathbf{T}_{Gj}\, \mathbf{q}_{Rj}, \tag{15}$$

$$\mathbf{q}_j = \begin{bmatrix} \mathbf{q}_R \\ \mathbf{q}_O \end{bmatrix}_s = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{V}_0 & \mathbf{T}_G \end{bmatrix}_s \begin{bmatrix} \tilde{\mathbf{q}}_O \\ \mathbf{q}_R \end{bmatrix}_s = \mathbf{T}_{Hj}\, \mathbf{q}_{Hj}, \tag{16}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{M}_{P1} \\ \mathbf{M}_{P2} & \mathbf{M}_{P3} \end{bmatrix}_j \begin{bmatrix} \ddot{\tilde{\mathbf{q}}}_O \\ \ddot{\mathbf{q}}_R \end{bmatrix}_j + \begin{bmatrix} 2\mathbf{\Delta} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_P \end{bmatrix}_j \begin{bmatrix} \dot{\tilde{\mathbf{q}}}_O \\ \dot{\mathbf{q}}_R \end{bmatrix}_j + \begin{bmatrix} \mathbf{\Omega}_0^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_P \end{bmatrix}_j \begin{bmatrix} \tilde{\mathbf{q}}_O \\ \mathbf{q}_R \end{bmatrix}_j = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}_j \tag{17}$$

$$\mathbf{M}_{Rj}\, \ddot{\mathbf{q}}_{Rj} + \mathbf{B}_{Rs}\, \dot{\mathbf{q}}_{Rj} + \mathbf{K}_{Rj}\, \mathbf{q}_{Rj} = \mathbf{f}_{Rj}, \quad \mathbf{M}_{Rj} = \mathbf{M}_{P3j},\ \mathbf{B}_{Rj} = \mathbf{B}_{Pj}, \quad \mathbf{K}_{Rj} = \mathbf{K}_{Pj}, \tag{18}$$

where $\mathbf{V}_{0j}$, $\mathbf{\Delta}_j$, $\mathbf{\Omega}_0{}^2{}_j$ are modal-spectral properties of the j^{th} substructures and matrix $\mathbf{T}_{Gj}$ results from the Guayan reduction method (11). The modal properties of the substructure are commonly obtained experimentally or on the basis of modal analysis of the FEM model of the relatively complex structure. Penetration of the common coordinates of individual substructures into one global vector $\mathbf{q}$ can later form the complete structure of the model. In this way it is possible to significantly reduce the DOF of the systems FEM model without compromising its dynamic properties.

4 Design of the added structural element by method of modal synthesis

Often times it is necessary to change the dynamic p properties of an existing structure. If the modal properties are known, the modal synthesis method can be used to obtain the desired modal – spectral properties. In such a case, the modal properties of the original system are virtually realized, for example, through FEM modeling of the added substructures in such a way that the modal properties of the modified system change in the desired way. The added substructures must be described through the same coordinates of the original (existing) structure, for which any of the above mentioned methods for reduction can be used and thus ensure the connection between the original and added structures [7], [9], [12].
If the existing proportionally damped system with the measured modal – spectral properties $\mathbf{V}_{0P}$, $\mathbf{\Omega}_{0P}$, $2\mathbf{\Delta}_{0P}$, is added to by the substructure represented by the reduced (to common coordinates) FEM model with coefficient matrices $\mathbf{M}_A$, $\mathbf{K}_A$, $\mathbf{B}_A$, where the damping matrix $\mathbf{B}_A$ is also proportional, the resulting system obtains the modal – spectral properties $\mathbf{V}_{0L}$, $\mathbf{\Omega}_{0L}$, $2\mathbf{\Delta}_{0L}$. Usually the added component is located in a defined area and not evenly across the original structure, the resulting structure then becomes non-proportionally damped.

4.1 Modal analysis of the non-porportionally damped system

To determine the modal and frequency properties of the disproportionately damped system, the system from n dimensional space represented by coefficients matrices **M, B, K** of order n is transformed to a system of $2n$ dimensional space with the coefficient matrices **N**, **P** [15] , [16] , [17]. Then

$$\mathbf{N}\,\dot{\mathbf{x}} - \mathbf{P}\mathbf{x} = \mathbf{r}, \tag{19}$$

where

$$\mathbf{P} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{B} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \in R^{2n,2n}, \ \ \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}, \mathbf{r} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \in R^{2n}. \tag{20}$$

From the solution of the eigenvalues for the symmetric coefficient matrix it also applies:

$$(\mathbf{P} - s_j\,\mathbf{N})\,\mathbf{w}_j = \mathbf{0}, \tag{21}$$

$$(\mathbf{K} + s_j\,\mathbf{B} + s_j^2\,\mathbf{M})\,\mathbf{v}_j = \mathbf{0}, \tag{22}$$

where vector $\mathbf{v}_j$ represents its j^{th} Eigen mode and $s_j = -\,\delta_j \pm i\ \omega_{Dj}$ is its j^{th} eigenvalue (assuming subcritical damping of the system). The real part of the eigenvalues represent the damping properties and the imaginary part represents the frequency properties. The conditions for orthonormality can be expressed by:

$$\mathbf{W}^{\mathrm{T}}\,\mathbf{P}\,\mathbf{W} = \mathbf{S},\ \mathbf{W}^{\mathrm{T}}\,\mathbf{N}\,\mathbf{W} = \mathbf{I}, \tag{23}$$

$$\mathbf{S}\,\mathbf{V}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{V}\,\mathbf{S} - \mathbf{V}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{V} = \mathbf{S},\ \mathbf{V}^{\mathrm{T}}\,\mathbf{B}\,\mathbf{V} + \mathbf{V}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{V}\,\mathbf{S} + \mathbf{S}\,\mathbf{V}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{V} = \mathbf{I} \tag{24}$$

where for the spectral matrix **S** and modal matrix **W** and **V**:

$$\mathbf{S} = diag(s_j) \in C^{,2n}, \quad \mathbf{W} = \{\mathbf{w}_j\} = \begin{bmatrix} \mathbf{V} \\ \mathbf{V}\,\mathbf{S} \end{bmatrix} \in \boldsymbol{C}^{2n,2n},\ \mathbf{V} = \{\mathbf{v}_j\} \in C^{n,2n}. \tag{25}$$

In cases where the substructure with the proportional damping matrix is for example $\mathbf{B} = \alpha\,\mathbf{M} + \beta\,\mathbf{K}$, the solution of the eigenvalues is simplified to [16]:

$$(\mathbf{K} - \mathbf{M}\ \omega_{0j}^2)\,\mathbf{v}_{0j} = \mathbf{0}, \quad \mathbf{K}\,\mathbf{V}_0 - \mathbf{M}\,\mathbf{V}_0\,\boldsymbol{\Omega}_0^2 = \mathbf{0}, \tag{26}$$

$$\begin{gathered}\boldsymbol{\Omega}_0 = diag(\omega_{0j}) \in R^n, \quad \boldsymbol{\Delta} = diag(\delta_j) \in R^n,\ \mathbf{V}_0 = \{\mathbf{v}_{0j}\} \in R^{n,n}, \\ \mathbf{V}_0^{\mathrm{T}}\,\mathbf{K}\,\mathbf{V}_0 = \boldsymbol{\Omega}_0^2,\ \mathbf{V}_0^{\mathrm{T}}\,\mathbf{M}\,\mathbf{V}_0 = \mathbf{I},\ \mathbf{V}_0^{\mathrm{T}}\,\mathbf{B}\,\mathbf{V}_0 = 2\,\boldsymbol{\Delta} = 2\,(\alpha\,\mathbf{I} + \beta\,\boldsymbol{\Omega}_0^2).\end{gathered} \tag{27}$$

The diagonal matrix $\boldsymbol{\Omega}_0$ contains elements of natural angular frequencies of the non-damped system ω_{0j} and the matrix of constant decay $\boldsymbol{\Delta}$ contains elements of constant decay for a proportionally damped system $\delta_j = 2\ \xi\ \omega_{0j}$, where ξ is the comparative damping of the analyzed substructure. The modal matrix $\mathbf{V}_0$ is composed of vectors $\mathbf{v}_{0j}$ representing the Eigen modes of this system. For the eigenvalues s_j and spectral matrix $\mathbf{S}$ of the proportional system the following applies:

$$s_j = -\delta_j \pm i\,(\omega_{0j}^2 - \delta_j^2)^{0,5} = -\delta_j \pm i\,\omega_{Dj}, \tag{28}$$

$$\mathbf{S} = -\boldsymbol{\Delta} \pm i\,(\boldsymbol{\Omega}_0^2 - \boldsymbol{\Delta}^2)^{0,5} = -\boldsymbol{\Delta} \pm i\,(\boldsymbol{\Omega}_0^2 - \boldsymbol{\Delta}^2)^{0,5} = -\boldsymbol{\Delta} \pm i\,\boldsymbol{\Omega}_D. \tag{29}$$

4.2 Modal synthesis of non-proportionally damped system

The above mentioned relations can be used to determine the modal and spectral properties of the original substructures modified by means of added substructures. The modal and spectral properties of the original proportionally damped system $\mathbf{V}_{0P}$, $\boldsymbol{\Omega}_{0P}$, $2\,\boldsymbol{\Delta}_{0P}$ are determined experimentally. The added, also proportionally damped system, is represented by the FEM model with coefficient matrices $\mathbf{M}_A$, $\mathbf{B}_A$, $\mathbf{K}_A$. These are composed so that the FEM model of the added structure reduces to the measured coordinates and forms elements of the matrix that are responsible for the measured coordinates. Elements of this matrix contain coordinates for which the added structures do not belong because it is localized only at predetermined areas equal to 0. This way, the experimentally and analytically obtained matrices have the same dimension.

The properties of such a modified system represent the modal and spectral matrix $\mathbf{V}_L$ and $\mathbf{S}_L$. From the conditions of orthonormality affected from the modal analysis of the modified system in *2n* space (23), it follows that:

$$[\mathbf{V}_L^T, \mathbf{S}_L\,\mathbf{V}_L^T] \begin{bmatrix} -(\mathbf{K}_P + \mathbf{K}_A) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_P + \mathbf{M}_A \end{bmatrix} \begin{bmatrix} \mathbf{V}_L \\ \mathbf{V}_L\,\mathbf{S}_L \end{bmatrix} = \mathbf{S}_L \tag{30}$$

$$[\mathbf{V}_L^T, \mathbf{S}_L\,\mathbf{V}_L^T] \begin{bmatrix} \mathbf{B}_P + \mathbf{B}_A & \mathbf{M}_P + \mathbf{M}_A \\ \mathbf{M}_P + \mathbf{M}_A & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_L \\ \mathbf{V}_L\,\mathbf{S}_L \end{bmatrix} = \mathbf{I}, \tag{31}$$

If transformation is used

$$\mathbf{V}_L = \mathbf{V}_{0P}\,\mathbf{T}_L, \tag{32}$$

$$[\mathbf{T}_L^T, \mathbf{S}_L\,\mathbf{T}_L^T] \begin{bmatrix} -(\boldsymbol{\Omega}_{0P}^2 + \mathbf{V}_{0P}^T\,\mathbf{K}_A\,\mathbf{V}_{0P}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} \end{bmatrix} \begin{bmatrix} \mathbf{T}_L \\ \mathbf{T}_L\,\mathbf{S}_L \end{bmatrix} = \mathbf{S}_L \tag{33}$$

$$[\mathbf{T}_L^T, \mathbf{S}_L\,\mathbf{T}_L^T] \begin{bmatrix} 2\boldsymbol{\Delta}_P + \mathbf{V}_{0P}^T\,\mathbf{B}_A\,\mathbf{V}_{0P} & \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} \\ \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{T}_L \\ \mathbf{T}_L\,\mathbf{S}_L \end{bmatrix} = \mathbf{I} \tag{34}$$

These relations represent conditions for orthonormality of the transformed system with spectral matrix $\mathbf{S}_L$ and modal matrix:

$$\mathbf{W}_T = \{\mathbf{w}_{Tj}\} = \begin{bmatrix} \mathbf{T}_L \\ \mathbf{T}_L\,\mathbf{S}_L \end{bmatrix}. \tag{35}$$

and with the coefficient matrices

$$\mathbf{P}_T = \begin{bmatrix} -(\mathbf{\Omega}_{0P}^2 + \mathbf{V}_{0P}^T\,\mathbf{K}_A\,\mathbf{V}_{0P}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} \end{bmatrix}, \tag{36}$$

$$\mathbf{N}_T = \begin{bmatrix} 2\mathbf{\Delta}_P + \mathbf{V}_{0P}^T\,\mathbf{B}_A\,\mathbf{V}_{0P} & \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} \\ \mathbf{I} + \mathbf{V}_{0P}^T\,\mathbf{M}_A\,\mathbf{V}_{0P} & \mathbf{0} \end{bmatrix}. \tag{37}$$

$\mathbf{P}_T$ and $\mathbf{N}_T$ are known elements of the coefficient matrix. the unknown spectral and modal matrices $\mathbf{S}_L = diag(s_{Lj})$ and $\mathbf{W}_T$ can be determined by solving the eigenvalues in the form (21):

$$(\mathbf{P}_T - s_{Lj}\,\mathbf{N}_T)\,\mathbf{w}_{Tj} = \mathbf{0}. \tag{38}$$

Directly from this the modal matrix of the modified system $\mathbf{S}_L$ emerges and therefore obtaining the modal matrix of the modified system $\mathbf{V}_L$ can be achieved using the transformation relation in (32)

4.3 Modal synthesis of the non-proportionately damped layered beam

The above mentioned modal synthesis method can be used also in determining the modal and spectral properties of beam structures with added layers of vibroisolation at specific points. A very simple illustration of such a system can be explained on an existing cantilever beam with a connected (added) beam which creates the modified beam seen in Fig. 4. also shows the schematic illustration of the above mentioned modal synthesis methodology.

From the schematic, coefficient matrices $\mathbf{M}_N$, $\mathbf{K}_N$ of the added beam reduces to the measurement coordinates y_2, y_3, which are shared with the original beam. The reduced coefficient matrices $\mathbf{M}_R$, $\mathbf{K}_R$ obtained in this way create a non-zero sub matrix expanded by the coefficient matrices $\mathbf{M}_A$, $\mathbf{K}_A$ of the added beam such, that it also represents the extra unmeasured coordinates y_1, y_4. This way the matrices have the same dimension as those measured by natural angular frequency, constant decay and with the modal matrix $\mathbf{\Omega}_{0P}$, $\mathbf{\Delta}_P$, $\mathbf{V}_{0P}$.

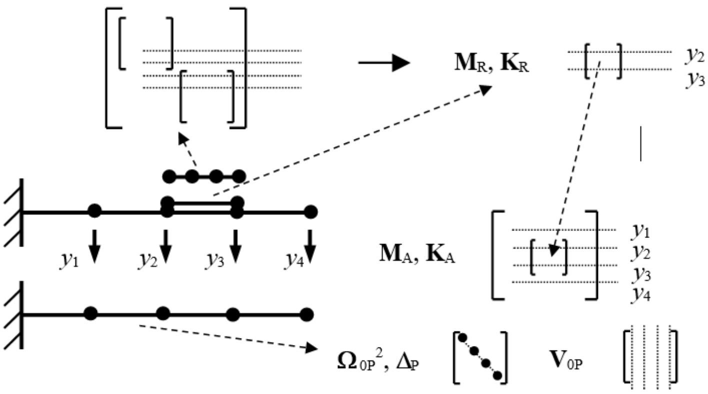


Fig. 4. Schematic methodology of modal synthesis.

The presented method can be automated and used for the parameter optimization of vibroisolating layers (orientation, geometry, material properties, etc...) [14]. The schematic representation of optimizing position *a* and thickness *h* of the vibroisolating layer with respect to the maximum ratio of damping ξ in the second Eigen mode is shown in Fig. 5. (c). From this dependence it is obvious that by increasing the thickness of the isolating layer, the ratio of damping also gradually increases. More complex characteristics have values of proportional damping dependent on the location of the isolating layer, which directly depends on the corresponding eigen modes in the presented system. Fig. 6. depicts the dependence of proportional damping of the 3rd and 4th modes and the position of the vibroisolating layer. The steel beam with an aluminium foam layer was considered (Tab. 1 and Tab. 2 – 500kg/m^3).

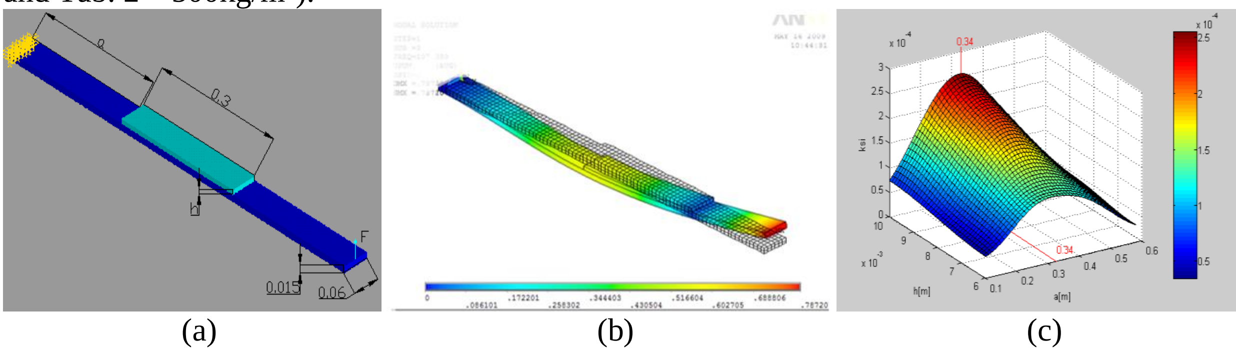


Fig. 5. Hierarchy (left to right) for the optimization of position “a” and thickness “h” of the vibro-isolating layer.

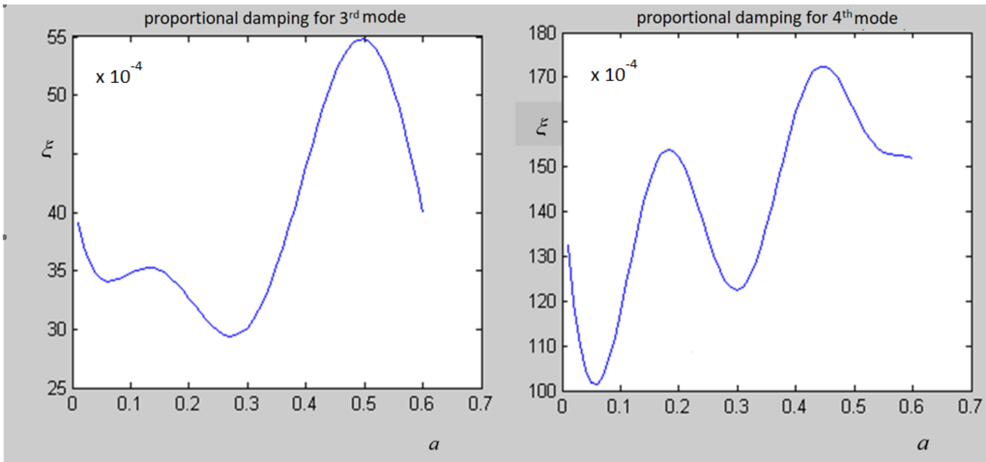
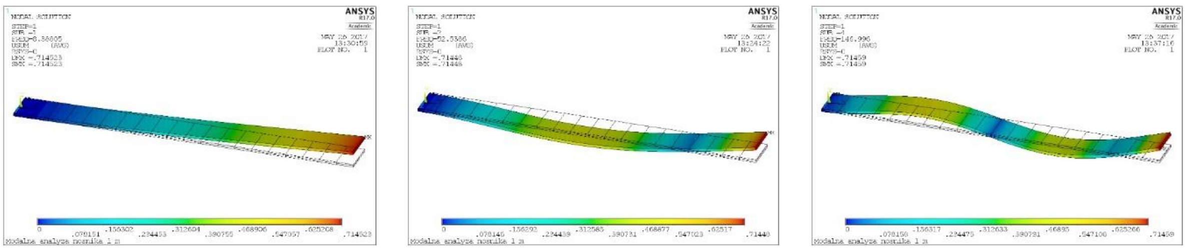


Fig. 6. Dependence of the proportional damping (3rd and 4th modes) on the position of the vibroisolating layer.

From the presented properties, it can be seen how it is possible to achieve the desired damping of some Eigen modes by choosing the appropriate position and thickness of the vibroisolating layer. Therefore, on the basis of modal synthesis it is possible, without any time-consuming calculations, to determine how effective the chosen parameters of the vibroisolating layer are and in what manner do the dynamic properties of the modifies structure change.

4.4 Structural dynamic modification of the beam by aluminium foams demonstrated on FE model

Structural modification of the beam by aluminium foam layers causes the change of dynamic parameters of the original structure, which is known as well as structural dynamic modification. It is possible to tune natural frequencies of the beam structure by adding one or more additive aluminium layers on suitable places due to which natural frequency change is required. Effective natural frequency tuning can be done by adding of modifying structure to places of anti-nodes of the appropriate mode shape. Desired dynamic parameters of the beam structure can be reached using aluminium layers with corresponding parameters that can be calculated by mentioned modal synthesis method. Next figure shows first three natural frequencies of bending vibration of the cantilever beam.



8.3881 Hz 52.539 Hz 147.00 Hz

Fig. 7. First three natural frequencies and mode shapes.

The picture bellow demonstrates structural dynamic modifications of the mentioned beam to reach desire modal and spectral properties.

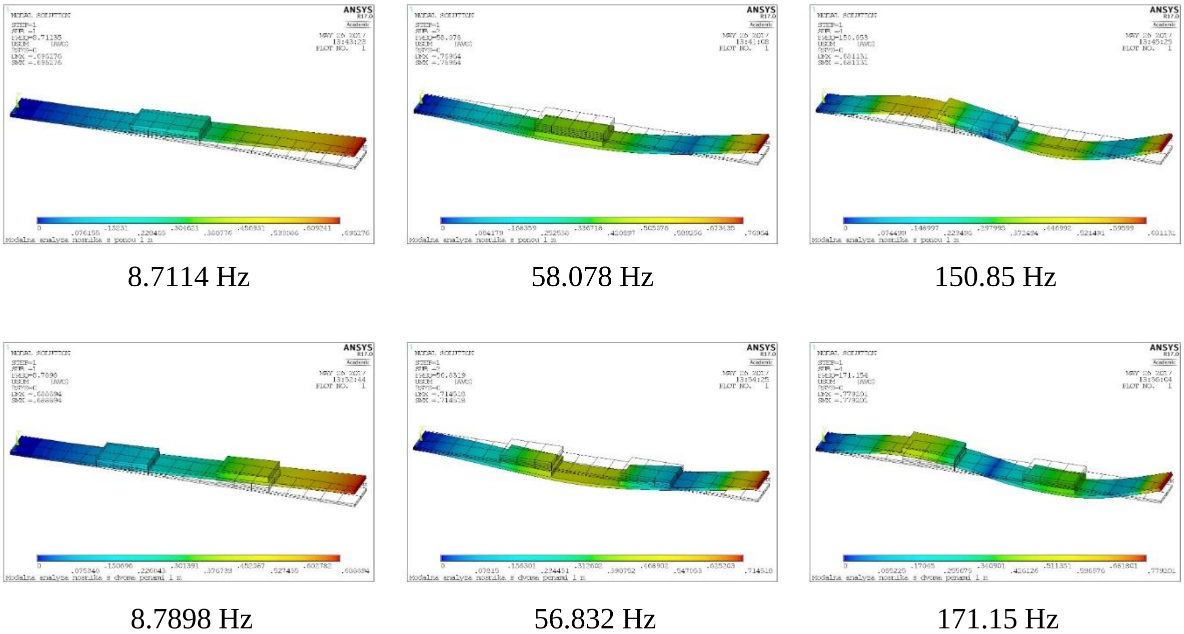


Fig. 8. Structural dynamic modifications of the beam to change its natural frequencies.

As is shown in pictures, modification with the one layer mostly influenced the second mode shape, while the two-layer modification is appropriate for the increasing of the third natural frequency.

5 Conclusion

This work presents methodology which allows for the purposeful change in dynamic properties of existing machine structures by adding additional components. This method is based on modal synthesis and is particularly effective in cases where it is not possible to obtain the reliable and standalone FEM model of the structure but measured modal properties of the system are available. At the same time, it is considered that the added components can be modelled utilizing FEM. Based on modal synthesis, it is possible to determine the modal properties with added component, using the methodology presented in this paper.
The presented methodology is applied to a situation where the added component is a layer of aluminium foam and is illustrated on a beam of a structure.

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