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SOME ALGEBRAIC PROPERTIES OF TENSORS

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Abstract. This paper is devoted to the study of some algebraic properties of tensors of type (1,3) which have the properties similar to Riemannian tensor on manifolds with affine connection. The investigations are related to certain decompositions of these tensors. The criteria are connected with systems of coordinates and some components of the tensors mentioned above.

Keywords: algebraic properties, tensors, Riemannian tensor

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1 Introduction

In the works by I. P. Egorov [1] and N. S. Sinyukov [9] were used criteria of spaces with non constant curvature connected with existence of non zero components of Riemannian tensor in certain special coordinates. These authors applied the criteria for studying of motions and geodesic mappings of Riemannian spaces and spaces with affine connection. The above mentioned criteria were formulated in the monograph by I. A. Schouten and J. J. Struik [8].

These criteria have been developed in the paper [6] by J. Mikeš and D. Moldobayev. Moreover, this problem is closely connected with decompositions of tensors, see [2, 4, 3].

This study is focused on detailed description of these criteria for tensors which have the algebraic properties of Riemannian tensor.

2 Tensors which have the algebraic properties of Riemannian tensor

Let manifolds M is assigned on some of coordinate neighborhood with coordinates $x^1, x^2, \ldots, x^n$ and let in M we determine the tensor T^h_{ijk} which has the algebraic properties of Riemannian tensor. We can write

$$T^h_{ijk} + T^h_{ikj} = 0, \quad (1)$$

$$T^h_{ijk} + T^h_{jki} + T^h_{kji} = 0. \quad (2)$$

These properties (1) and (2) have, for example [4, 7, 9], *Weyl tensor of projective curvature* on spaces with affine connection

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n+1}\,\delta^h_i (R_{jk} - R_{kj}) + \frac{1}{(n+1)\,(n-1)}\,[(nR_{ij} + R_{ji})\,\delta^h_k - (nR_{ik} + R_{ki})\,\delta^h_j)],$$

where R^h_{ijk} and $R_{ik} = R^\alpha_{i\alpha k}$ are the Riemannian and Ricci tensors, respectively. Weyl tensor of projective curvature in equiaffine spaces (for which $R_{ik} = R_{kj}$) has the following simply form

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}\,(R_{ij}\,\delta^h_k - R_{ik}\,\delta^h_j),$$

and, for $n > 2$, *Weyl tensor of conformal curvature* on (pseudo-) Riemannian spaces

$$C^h_{ijk} = R^h_{ijk} - \delta^h_j L_{ik} + \delta^h_k L_{ij} - L^h_j g_{ik} + L^h_k g_{ij},$$

where

$$L_{ij} = \frac{1}{n-2}\,(R_{ij} - \frac{R}{2(n-1)}\,g_{ij})$$

is the Brinkmann tensor, $L^h_i = g^{h\alpha} L_{\alpha i}$, $R = R_{ij} g^{ij}$ is a scalar curvature, g_{ij} is a metric tensor, and g^{ij} are components of its inverse matrices.

These properties yield also for *Yano tensor of concircular curvature*

$$Y^h_{ijk} = R^h_{ijk} + \frac{R}{n\,(n+1)}\,(g_{ij}\,\delta^h_k - g_{ik}\,\delta^h_j).$$

3 Tensor transformation law

During the transformation of the coordinate system

$$x'^h = x'^h(x^1, x^2, \dots, x^n),$$

components of tensor T are changed according to the tensor law

$$T'^h_{ijk}(x') = T^\alpha_{\beta\gamma\delta}(x)\,A^h_\alpha\,B^\beta_i\,B^\gamma_j\,B^\delta_k, \tag{3}$$

where

$$A^h_i = \frac{\partial x'^h}{\partial x^i}$$

and $\|B\|$ is inverse matrix of $\|A\|$.

Furthermore, the same is valid for a given point on these spaces. Here the tensor T is considered to be given in this point. Hence, the formulae analogous to (3) is valid.

The components of tensor T are changed according to the law

$$T'^h_{ijk} = T^\alpha_{\beta\gamma\delta}\,A^h_\alpha B^\beta_i\,B^\gamma_j\,B^\delta_k, \tag{4}$$

where A^h_i is real regular matrix, and $\|B\|$ is inverse matrix of $\|A\|$.

These changes of tensors components satisfy linear transformations

$$x'^h = A^h_i\,x^i.$$

4 The Main Theorem of special tensors decomposition

We proved the following Main Theorem.

Theorem 1 *Let in each coordinates is valid one of the conditions*

$$a) \qquad T^H_{IIK} = 0 \tag{5}$$

or

$$b) \qquad T^H_{IKM} = 0 \tag{6}$$

where H, I, K, M are fixed commonly different indices.

Then tensor T has the following decomposition

$$T^h_{ijk} = \delta^h_i\,(p_{jk} - p_{kj}) + \delta^h_k\,p_{ji} - \delta^h_j\,p_{ki} \tag{7}$$

where

$$p_{kj} = \frac{n\,T_{jk} + T_{kj}}{n^2 - 1} \quad \textit{and} \quad T_{ij} = T^\alpha_{ij\alpha}.$$

Proof Let us assume that the conditions mentioned above in the Theorem are true. Firstly, we prove that formula (5) or (6) are true in each coordinates, so consequently, it follows that

$$a) \qquad T^h_{iik} = 0 \tag{8}$$

or

$$b) \qquad T^h_{ijk} = 0 \tag{9}$$

in each coordinates for any commonly different indices h, i, j, k.

Let us consider the following linear transformations

$$x'^p = x^p + r \cdot x^q, \quad x'^s = x^s, \quad s \neq p, \tag{10}$$

where r is a certain constant, p and q are commonly different fixed indices.

Then A^h_i and B^h_i in formula (4) have the following form $A^h_i = B^h_i = \delta^h_i$ and $A^p_q = -B^p_q = r$ for $h \neq p$ or $h \neq q$.

Let us compute the following components of tensor T in a new coordinate system determined by transformation (10)

$$T'^h_{pqk} = T^h_{pqk} - r \cdot T^h_{ppk}, \tag{11}$$

$$T'^h_{qqk} = T^h_{qqk} - r \cdot (T^h_{pqk} + T^h_{qpk}) + r^2 \cdot T^h_{ppk}, \tag{12}$$

$$T'^p_{qjk} = T^p_{qjk} + r \cdot (T^q_{qjk} - T^p_{pjk}) - r^2 \cdot T^q_{pjk}, \tag{13}$$

$$T'^p_{iqk} = T^p_{iqk} + r \cdot (T^q_{iqk} + T^p_{ipk}) - r^2 \cdot T^q_{ipk}, \tag{14}$$

$$T'^p_{qqk} = T^p_{qqk} + r \cdot (T^q_{qqk} - T^p_{pqk} - T^p_{qpk}) - r^2 \cdot (T^q_{pqk} + T^q_{qpk} - T^p_{ppk}) - r^3 \cdot T^q_{ppk}. \tag{15}$$

In the last formulas all different kind indices are not equivalent. Here and further by p and q no summing is provided.

First of all considering (11) it is easy to find out that from (9) follows (8).

Let us prove the contrary. Let the conditions (8) are valid, then $T^h_{iik} = 0$ in any coordinate system, under commonly different indices h, i, j. So from (14) we obtain $T^h_{ijk} + T^h_{jik} = 0$ where $h, i, j, k \neq$. The above mentioned indices are commonly different.

After alternation the last formula under j, k and considering the properties (1) and (2) of the tensor T we obtain that $T^h_{ijk} = 0$ for $h, i, j, k \neq$.

Finally we get

$$T^h_{ijk} = 0 \tag{16}$$

in any coordinate system for all indices $h, i, j, k \neq$.

Furthermore, from (13) and (14) we consequently obtain

$$T^q_{qjk} = T^p_{pjk} \quad \text{and} \quad T^q_{iqk} = T^p_{ipk} \tag{17}$$

for $p, q \neq j, k$.

Basing on (17), we can come to the following conclusion

$$T^p_{pjk} = A_{jk} \quad \text{and} \quad T^p_{ipk} = B_{jk} \tag{18}$$

for all indices p, j, k bat $p \neq j, k$, and A_{jk} and B_{jk} are certain geometric objects.

Considering (16) and (18) and that r, generaly speaking, is any constant, then from (15) we get

$$T^p_{ppk} = A_{pk} + B_{pk} \tag{19}$$

for any indices p and k, $p \neq k$.

From (16), (18), (19) we can write the following $T^h_{ijk} = \delta^h_i A_{jk} + \delta^h_j B_{ik} - \delta^h_k B_{ij}$, and by graduate contracting, taking into consideration the algebraic properties of tensor T, we get the formula (7). The theorem is proved.

5 Applications of the Main Theorem

Basic on Theorem 1 the following theorem is true.

Theorem 2 *Let in each coordinate system of space with affine connection A_n for the components of Riemannian tensor is valid one of the conditions*

$$a) \qquad R^H_{IIK} = 0 \quad \textit{or} \quad b) \qquad R^H_{IKM} = 0$$

where H, I, K, M are fixed commonly different indices. Then A_n is projectively Euclidean space.

Proof of this theorem follows from the structure of the curvature tensor R, which we obtain from the Main Theorem. Analog of Theorem 2 is true also for the Weyl tensor of projective curvature W^h_{ijk}, whereas, if we use the traceless property [3, 5].

This Theorem generalizes the results obtained by I. A. Schouten and D. J. Struik [8, p. 197] and shortens I. P. Egorov computing [1] assessment of the order of the complete groups of moving in Riemannian spaces and spaces with affine connections.

In (pseudo-) Riemannian spaces there are the following theorems

Theorem 3 *Let in each coordinate system of (pseudo-) Riemannian space V_n for the components of Riemannian tensor is valid one of the conditions $R^H_{IIK} = 0$ or $R^H_{IKM} = 0$, where H, I, K, M are fixed commonly different indices. Then V_n has constant curvature.*

Analog of Theorem 3 is true also for the Weyl tensor of projective curvature W^h_{ijk}, and for the Yano tensor of concircular curvature Y^h_{ijk}. Let as note that for dimension $n = 2$ these tensors are vanishing. The Weyl tensor of conformal curvature C^h_{ijk} is vanishing for dimension three. For dimension $n > 3$ tensor C^h_{ijk} is vanishing if and only if V_n is conformally (pseudo-) Euclidean. Because this tensor is traceless then, from mentioned above, the following theorem holds.

Theorem 4 *Let in each coordinate system of (pseudo-) Riemannian space V_n for the components of Riemannian tensor is valid one of the conditions $C^H_{IIK} = 0$ or $C^H_{IKM} = 0$, where H, I, K, M are fixed commonly different indices. Then V_n is conformally (pseudo-) Euclidean.*

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