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HANKEL TRANSFORM AND FREE VIBRATION OF A LARGE CIRCULAR MEMBRANE

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Abstract. Integral transforms are a powerful apparatus for solving initial value and boundary value problems for linear differential equations. Paper is primarily attended to Hankel integral transform and shows a utilization of the integral transforms for solving partial differential equation- the equation of vibration of circular membrane.

Keywords: integral transform, Hankel transform, partial differential equation

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„One cannot understand ... the universality of laws of nature, the relationship of thinngs, withouth an understanding of mathematics. There is no other way to do it.“

Richard P. Feynman

The theory of integral transorms in the classical sence can be traced back to about two hundred years. But the theory of distributions is only ninety five years old. Some elements of the theory of distributions may be found in [7]. The author actually tried to generalize to concept and classical operations to a larger field in which the Cauchy problem can be solved.

The papers [3],[4],[5] gave a firm footing to the theory of distributions. It has been rightly stated by a reputed mathematician that „twentieth century may appropriatety be called the century of Functional Analysis“. It may be pointed out that the theory of distributions occupies a very important role in the field of application of functional analysis. During the last seventy years the phenomenal growth in the theory of partial differential equations is the result of theory of distributions which has acted as a catalytic agent.

In [6], [8], [2], authors gave other approaches to distribution theory.

It all started with giving mathematically accepted definition to Dirac´s Delta function defined by

$$\delta(x) = 0 \text{ for } x \neq 0 \text{ and}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

This function was used to represent mathematically the concept of a shock, like the impact of hammer in mechanics or a large voltage of very small duration in electrical engineering. But it is not a function in the strict sence. For in the theory of integrals if a function is zero almost everywhere in an interval, its integral over that interval is also zero. Actually Delta function is a functional.

The Dirac delta function, $\delta(x)$ is defined so that for any good function $\Phi(x)$,

$$\int_{-\infty}^{\infty} \delta(x)\Phi(x) dx = \Phi(0).$$

Herman Hankel (1839-1873), a German mathematician, is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions which depend only on the distance from the origin. He also studied Bessel functions. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylidrical polar coordinates.

Bessel function

Function $J_v(x)$ is a solution of a Bessel equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0,$$

where $v \geq 0$ is called index of Bessel equation.

If $v = n$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x sin\theta)\, d\theta,$$

or $$J_n(x) = \frac{1}{2\pi}\int_{\phi_0}^{2\pi+\phi_0} e^{i(n\theta - x sin\theta)} d\theta \text{ [1].}$$

We introduce the definition of the *Hankel transform* from the two-dimensional Fourier transform and its inverse given by

$$\mathcal{F}\{f(x,y)\} = F(k,l) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\kappa,r)} f(x,y) dx\, dy, \quad (1)$$

$$\mathcal{F}^{-1}\{F(k,l)\} = f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\kappa,r)} F(k,l) dk\, dl, \quad (2)$$

where $\boldsymbol{r} = (x,y)$, $\boldsymbol{\kappa} = (k,l)$.

Introducing polar coordinates

$$x = r.cos\theta, \qquad k = \kappa.\cos\phi,$$

$$y = r.sin\theta, \qquad l = \kappa.\sin\phi,$$

where $r, \kappa \in \langle 0, \infty)$ a $\theta, \phi \in \langle 0, 2\pi)$ and $J = r$.

Than

$$\boldsymbol{r}.\boldsymbol{\kappa} = \kappa.r(\cos\phi\ cos\theta - \sin\phi\ sin\theta) = \kappa.rcos(\theta - \phi).$$

We denote $F(\kappa.\cos\phi, \kappa.\sin\phi) = G(\kappa, \phi)$,

where $G(\kappa, \phi) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{-i\kappa.rcos(\theta-\phi)} f(r,\theta) r\, d\theta dr$ (3)

We next assume $f(r,\theta) = e^{in\theta} f(r)$, which is not a very severe restriction, and make a change of variable $\theta - \phi = \alpha - \frac{\pi}{2}$ to reduce (3) to the form

$$G(\kappa, \phi) = \frac{1}{2\pi}\int_0^\infty \int_{\frac{\pi}{2}-\phi}^{2\pi+\frac{\pi}{2}-\phi} e^{-i\kappa.rcos\left(\alpha-\frac{\pi}{2}\right)} e^{in.rcos\left(\alpha+\phi-\frac{\pi}{2}\right)} f(r) r\, d\alpha dr.$$

Because $cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha$, than we have

$$G(\kappa, \phi) = e^{in\left(\phi-\frac{\pi}{2}\right)} \int_0^\infty \left[\frac{1}{2\pi}\int_{\frac{\pi}{2}-\phi}^{2\pi+\frac{\pi}{2}-\phi} e^{i(n\alpha - \kappa r\sin\alpha)} d\alpha\right] f(r) r dr.$$

Using the integral representation of the Bessel function of order *n*, we have

$$G(\kappa, \phi) = e^{in\left(\phi-\frac{\pi}{2}\right)} \int_0^\infty r J_n(\kappa r) f(r) r dr = e^{in\left(\phi-\frac{\pi}{2}\right)} \tilde{f}_n(\kappa),$$

where $\tilde{f}_n(\kappa)$ is called the *Hankel transform* of $f(r)$ defined formally by

$$\tilde{f}_n(\kappa) = \mathcal{H}_n\{f(r)\} = \int_0^\infty r J_n(\kappa r) f(r) dr.$$

Similarly, in terms of the polar variables with the assumption $f(x,y) = f(r,\theta) = e^{in\theta} f(r)$, $J = \kappa$, the inverse Fourier transform becomes

$$e^{in\theta} f(r) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} e^{i\kappa.rcos(\theta-\phi)} G(\kappa, \phi)\, \kappa d\phi d\kappa,$$

which is, by the change of variables $\theta - \phi = -\alpha - \frac{\pi}{2}$, we have

$$e^{in\theta} f(r) = \frac{1}{2\pi}\int_0^\infty \int_{-\theta-\frac{\pi}{2}}^{2\pi+\left(-\theta-\frac{\pi}{2}\right)} e^{i\kappa.rcos\left(-\alpha-\frac{\pi}{2}\right)} e^{in(\theta+\alpha)} \tilde{f}_n(\kappa)\, \kappa d\alpha d\kappa =$$

$$= e^{in\theta}\int_0^\infty \left[\frac{1}{2\pi}\int_{-\theta-\frac{\pi}{2}}^{2\pi+\left(-\theta-\frac{\pi}{2}\right)} e^{i(n\alpha - \kappa r sin\alpha}\, d\alpha\right] \tilde{f}_n(\kappa) \kappa d\kappa =$$

$$= e^{in\theta}\int_0^\infty \kappa J_n(\kappa r) \tilde{f}_n(\kappa) d\kappa.$$

Thus, the inverse Hankel transform is defined by

$$f(r) = \mathcal{H}_n^{-1}\{\tilde{f}_n(\kappa)\} = \int_0^\infty \kappa J_n(\kappa r) \tilde{f}_n(\kappa) d\kappa.$$

Free Vibration of a Large Circular Membrane

Obtain the solution of the free vibration of a large circular elastic membrane governed by the initial value problem

$$c^2\left(\frac{\partial^2 u(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,t)}{\partial r}\right)=\frac{\partial^2 u(r,t)}{\partial t^2}, 0<r<\infty, t>0, \tag{4}$$

$$u(r,0)=f(r), \qquad \left.\frac{\partial u(r,t)}{\partial t}\right|_{t=0}=g(r), \quad 0\le r<\infty \tag{5}$$

where $c^2=\frac{T}{\rho}$ =constant, T is the tension in the membrane, and ρ is the surface density of the membrane.

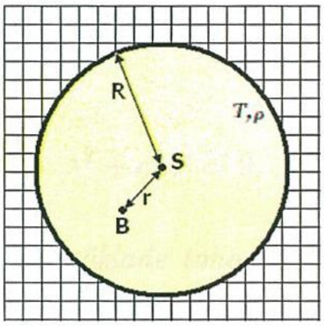

Fig. 1. Circular elastic membrane.

At the beginning, the membrane deflects from the equilibrium position.

Solution:

Equation (4) is a hyperbolic type, because for the matrix

$$\begin{pmatrix} c^2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda_1 = c^2, \qquad \lambda_2 = -1.$$

Application of the zero-order Hankel transform with respect to r

$$\tilde{u}(\kappa,t)=\int_0^\infty rJ_0(\kappa r)u(r,t)dr\,,$$

to (4) with conditions (5) gives

$$\frac{d^2\tilde{u}}{dt^2}=-c^2\kappa^2\tilde{u}(\kappa,t), \tag{6}$$

$$\tilde{u}(\kappa,0)=\tilde{f}(\kappa), \qquad \left.\frac{d\tilde{u}}{dt}\right|_{t=0}=\tilde{g}(\kappa) \tag{7}$$

We got a homogenne ordinar linear differential equation of second order with constant coeficients. Characteristic equation is

$$\lambda^2 + c^2\kappa^2 = 0,$$

where

$$\lambda_1 = ic\kappa\,, \qquad \lambda_2 = -ic\kappa.$$

Solution of equation (6) is in the form

$$\tilde{u}(\kappa, t) = c_1 \cos(c\kappa t) + c_2 \sin(c\kappa t).$$

From (7) we got, that $c_1 = \tilde{f}(\kappa)$, $c_2 = (c\kappa)^{-1}\tilde{g}(\kappa)$,

and $\tilde{u}(\kappa, t) = \tilde{f}(\kappa)\cos(c\kappa t) + (c\kappa)^{-1}\tilde{g}(\kappa)\sin(c\kappa t)$.

The inverse Hankel transformation leads to the solution

$$u(r,t) = \int_0^\infty \kappa J_0(\kappa r)\tilde{f}(\kappa)\cos(c\kappa t)\,d\kappa + \frac{1}{c}\int_0^\infty J_0(\kappa r)\tilde{g}(\kappa)\sin(c\kappa t)\,d\kappa\,. \qquad (8)$$

In particular, we consider

$$f(r) = Aa(r^2 + a^2)^{-\frac{1}{2}}, \quad a\,, A \in R, \quad g(r) = 0.$$

So that

$$\tilde{g}(\kappa) = \int_0^\infty rJ_0(\kappa r)0dr \equiv 0\,,$$

$$\tilde{f}(\kappa) = \int_0^\infty rJ_0(\kappa r)Aa(r^2 + a^2)^{-\frac{1}{2}}\,dr = \frac{Aa}{\kappa}e^{-a\kappa}\,.$$

From (8) we have

$$u(r,t) = \int_0^\infty \kappa J_0(\kappa r)\frac{Aa}{\kappa}e^{-a\kappa}\cos(c\kappa t)\,d\kappa = Aa\int_0^\infty J_0(\kappa r)e^{-a\kappa}\cos(c\kappa t)\,d\kappa =$$

$$= Aa\mathrm{Re}\int_0^\infty J_0(\kappa r)e^{-\kappa(a+ict)}d\kappa = Aa\,Re\left\{\int_0^\infty \kappa J_0(\kappa r)\frac{e^{-\kappa(a+ict)}}{\kappa}d\kappa\right\} = Aa\,Re\left\{\frac{1}{\sqrt{r^2+(a+ict)^2}}\right\}.$$

We will graphically illustrate the solution. The figure shows the vibration of the circular membrane with the center at the point $S = [0,0]$ and the radius $R = 4\,m$ from $t = 0\,s$ to $t = 20\,s$, where T = 2,43 $N\,m^{-1}$, and ρ = 4,29 $kg\,m^{-2}$, A = 4,5 , a = 1,3.

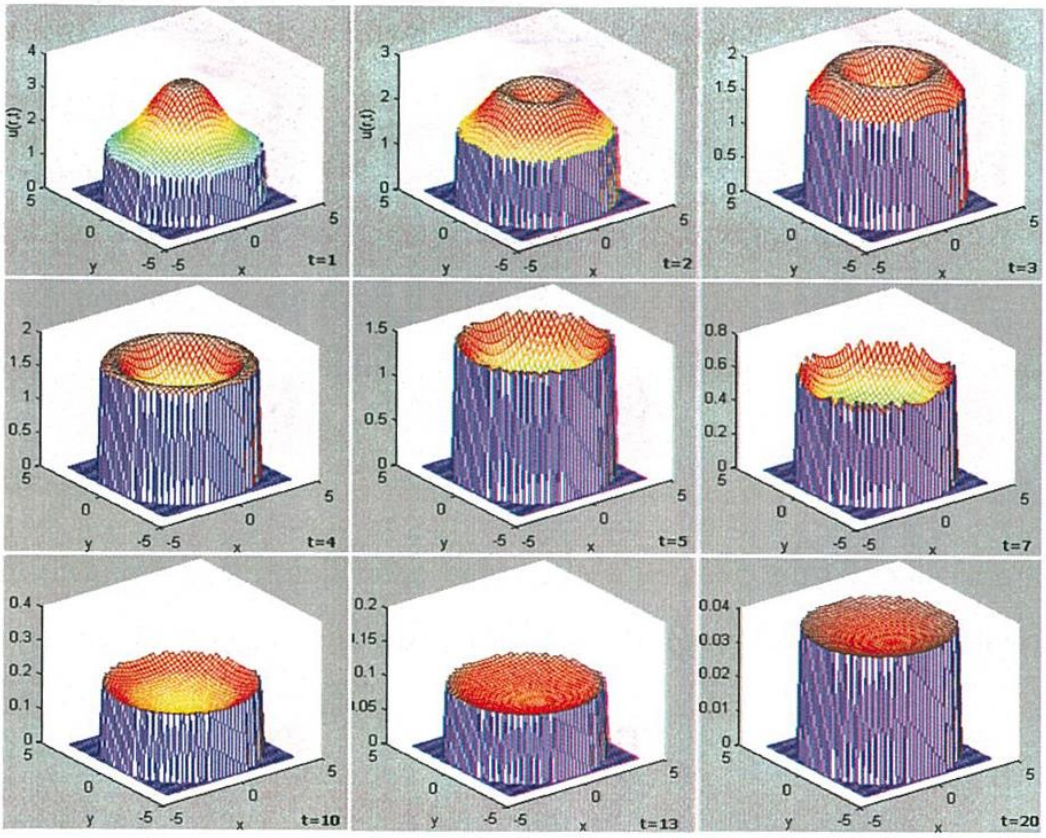

Fig. 2. Vibration of circular elastic membrane.

Conclusion

"A teacher can never truly teach unless he is still learning himself. A lamp can never light another lamp unless it continues to burn its own flame. The teacher who has come to the end of his subject, who has no living traffic with his knowledge but merely repeats his lessons to his students, can only load their minds, he cannot quicken them. "

Rabindranath Tagore

The importance of integral transforms is that they provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. Integral transforms have many mathematical and physical applications, their use is still predominant in advanced study and research. The paper was developed as a result of experience in a teaching advanced undergraduates students in mathematics and engineering. Main feature is a systematic mathematical treatment of the theory and method of integral transforms that gives the students a clear understanding of the subject and its varied applications.

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