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REGULARITY CRITERION FOR LOCAL IN TIME EXISTENCE OF STRONG SOLUTIONS TO THE NAVIER-STOKES EQUATIONS WITH NAVIER'S TYPE BOUNDARY CONDITIONS

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Abstract. In this contribution we deal with the system of the Navier-Stokes equations with boundary conditions of the Navier's type on the bounded smooth convex domain. The main theorem gives condition for local in time existence of strong solutions to this system. This result is modification of results of Robinson et al. for solutions of the Navier-Stokes system on the whole space or on the space-periodic domains.

Keywords: Navier-Stokes equations, Navier's type boundary conditions, regularity

Mathematics subject classification: Primary 35Q30, 35B35; Secondary 76D05, 76E09.

1 Introduction

Let $\beta > 0$, $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega$ of the class $C^{2+\beta}$. We denote $Q_T = \Omega \times (0, T)$. We study the following initial–boundary value Navier–Stokes problem

$$\partial_t \boldsymbol{u} + \nu\, \mathbf{curl}^2 \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla P = \mathbf{0} \qquad \text{in } Q_T, \tag{1.1}$$

$$\mathbf{div}\, \boldsymbol{u} = 0 \qquad \text{in } Q_T, \tag{1.2}$$

$$\text{a) } \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \text{b) } \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

$$\boldsymbol{u}(\,.\,, 0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{1.4}$$

Here, $\boldsymbol{u} = (u_1, u_2, u_3)$ and P denote the velocity of motion and the associated pressure. By $\boldsymbol{n}$ and ν we denote the outer normal vector field on $\partial\Omega$ and the kinematic coefficient of viscosity, respectively. For simplicity we suppose $\nu = 1$. Equations (1.1), (1.2) describe the motion of a viscous incompressible fluid in domain Ω.

H. Navier formulated so called Navier's boundary conditions in 1824. They takes the form

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad [\mathbb{T} \cdot \boldsymbol{n}]_\tau + \gamma\, \boldsymbol{u} = 0. \tag{1.5}$$

The symbols $\mathbb{T}$ and the subscript τ denote the stress tensor and the tangential component, respectively. The first Navier's condition is the condition of impermeability of the wall. The second condition expresses the requirement that the tangential component of the stress, with which the fluid acts on the boundary, is proportional to the velocity. The stress tensor has the form $\mathbb{T} = -p\mathbb{I} + \nu\,[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$. Consequently, the second Navier condition (1.5b) can be written in the form

$$\nu\,\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} - 2\nu\,\boldsymbol{u} \cdot \nabla \boldsymbol{n} + \gamma\,\boldsymbol{u} \;=\; 0.$$

If we suppose that coefficient γ of friction between the fluid and the boundary is zero (the case of the so called perfect slip) and the curvature of the wall is neglected then (1.5) is equivalent to our boundary condition (1.3b). The Navier–Stokes equations with the boundary conditions (1.3) are studied, e.g., in papers [1], [2], [3], [4], [13], [16].

1.1 Notation of function spaces and operators.

We denote vector–valued functions and spaces of such functions by boldface letters. We use these function spaces and operators:

- c is a generic constant, i.e. a constant whose value may vary from line to line. On the other hand, numbered constants have fixed values throughout the whole paper.
- $\mathbf{C}^{\infty}_{0,\sigma}(\mathbb{R}^3) = \{\boldsymbol{\phi} \in \mathbf{C}^{\infty}(\Omega);\ supp\,\phi \subset \Omega,\ \mathbf{div}\,\phi \equiv 0\}$.
- $\mathbf{L}^s_{\sigma}(\Omega)$ is a closure of $\mathbf{C}^{\infty}_{0,\sigma}(\Omega)$ in $\mathbf{L}^s(\Omega)$.
- The Hemholtz projection of $\mathbf{L}^s_{\sigma}(\Omega)$ onto $\mathbf{L}^s(\Omega)$ is denoted by $\mathbf{P}_{\sigma}$. The following holds

$$\mathbf{L}^s(\Omega) = \mathbf{L}^s_{\sigma}(\Omega) \oplus G(\Omega) \tag{1.6}$$

  where $G(\Omega) = \{\nabla p;\ p \in \mathbf{W}^{1,s}(\Omega)\}$, see, e.g., [6] for the proof.
- $\mathbf{L}^2_{\sigma}(\Omega)^{\perp} = \{\nabla\varphi;\ \varphi \in W^{1,2}(\Omega)\}$ (The symbol $\mathbf{L}^2_{\sigma}(\Omega)^{\perp}$ denotes the orthogonal complement to $\mathbf{L}^2_{\sigma}(\Omega)$ in $\mathbf{L}^2(\Omega)$).
- $\mathbf{W}^{1,2}_{\sigma}(\Omega) := \mathbf{W}^{1,2}(\Omega) \cap \mathbf{L}^2_{\sigma}(\Omega)$. (Functions in this space are divergence–free in Ω and have the normal component on $\partial\Omega$ equal to zero in the sense of traces.)
- $\|\,.\,\|_k$ (respectively $\|\,.\,\|_{l,k}$) denotes the L^k–norm (respectively the $W^{l,k}$–norm) of a scalar–valued or vector–valued or tensor–valued function in Ω. The scalar product in $\mathbf{L}^2(\Omega)$ is denoted by $(\,.\,,\,.\,)_2$.
- The dual space to $\mathbf{W}^{1,2}_{\sigma}(\Omega)$ is denoted by $\mathbf{W}^{-1,2}_{\sigma}(\Omega)$. The duality between the elements of $\mathbf{W}^{-1,2}_{\sigma}(\Omega)$ and $\mathbf{W}^{1,2}_{\sigma}(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle_{\Omega}$ and the norm in $\mathbf{W}^{-1,2}_{\sigma}(\Omega)$ is denoted by $\|\cdot\|_{-1,2}$.
- $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega) := \big\{\boldsymbol{u} \in \mathbf{W}^{2,2}(\Omega) \cap \mathbf{W}^{1,2}_{\sigma}(\Omega);\ \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\big\}$. (The subscript „nc" indicates the Navier–type boundary conditions.)
- $\mathbf{H}(\Omega)$ is the space of all divergence–free functions $\boldsymbol{u} \in \mathbf{W}^{1,2}(\Omega)$ such that $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$.
- $\mathbf{T}_1 := \mathbf{curl}$ is the operator with the domain $D(\mathbf{T}_1) = \mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$.
- $\mathbf{T}_2 := \mathbf{curl}$ is the operator with the domain $D(\mathbf{T}_2) = \mathbf{H}(\Omega)$. The kernel of both the operators $\mathbf{T}_1$ and $\mathbf{T}_2$ is trivial because domain Ω is simply connected.

- $\mathbf{S} := \mathbf{T}_2 \circ \mathbf{T}_1$ is an operator in in $\mathbf{L}^2_\sigma(\Omega)$. $\mathbf{S}$ is one of the concrete realizations of the so called Stokes operator. The domain of $\mathbf{S}$ is $D(\mathbf{S}) = D(\mathbf{T}_1) = \mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$.

Below we list some properties of operators $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{S}$, see [13] and [16].

- $\mathbf{T}_1$ is a linear bijective operator from $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$ onto $\mathbf{H}(\Omega)$. The inverse operator is bounded as an operator from $\mathbf{W}^{1,2}(\Omega)$ into $\mathbf{W}^{2,2}(\Omega)$.
- $\mathbf{T}_2$ is a linear bijective operator from $\mathbf{H}(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$. The inverse operator is bounded as an operator from $\mathbf{L}^2(\Omega)$ into $\mathbf{W}^{1,2}(\Omega)$.
- Operator $\mathbf{S}$ is a linear bijective operator from $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$. The inverse operator $\mathbf{S}^{-1} = \mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$ is a bounded operator from $\mathbf{L}^2_\sigma(\Omega)$ onto $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$ and $\mathbf{S} = -\Delta$.
- Operator $\mathbf{S}$ commutes with the projection $\mathbf{P}_\sigma$, i.e.

$$\mathbf{P}_\sigma \mathbf{S} = \mathbf{S}\mathbf{P}_\sigma \boldsymbol{u}$$

  for $\boldsymbol{u} \in D(\mathbf{S})$. Consequently

$$-\mathbf{P}_\sigma \Delta = -\Delta. \tag{1.7}$$

- Operator $\mathbf{S}$ is positive and selfadjoint in $\mathbf{L}^2_\sigma(\Omega)$. The eigenvalues of $\mathbf{S}$ form a non–decreasing sequence $\{\lambda_i\}$ of positive real numbers and they have the same algebraic and geometric multiplicity. Corresponding eigenfunctions $\{\mathbf{e}^i\}$ can be chosen so that they form a complete orthonormal system in $\mathbf{L}^2_\sigma(\Omega)$.
- Although the operator $\mathbf{S}^{1/2}$ is different from $\mathbf{T}_1$, one can easily check the identities: $\|\mathbf{S}^{1/2}\boldsymbol{u}\|_2^2 = (\mathbf{S}\boldsymbol{u}, \boldsymbol{u})_2 = \|\mathbf{curl}\,\boldsymbol{u}\|_2^2 = \|\mathbf{T}_1\boldsymbol{u}\|_2^2$ for $\boldsymbol{u} \in D(\mathbf{S})$. Consequently, there exist positive constants c_1 and c_2 so that for $\boldsymbol{u} \in D(\mathbf{S})$, we have

$$c_1\,\|\mathbf{S}^{1/2}\boldsymbol{u}\|_2 \;\le\; \|\boldsymbol{u}\|_{1,2} \;\le\; c_2\,\|\mathbf{S}^{1/2}\boldsymbol{u}\|_2. \tag{1.8}$$

  Inequalities (1.8) show that the norms $\|\mathbf{S}^{1/2}.\,\|_2$ and $\|\,.\,\|_{1,2}$ are equivalent in $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$. Using the Friedrichs–type inequality

$$\|\boldsymbol{u}\|_2 \;\le\; C\,\|\nabla\boldsymbol{u}\|_2 \tag{1.9}$$

  (where $C = C(\Omega)$), satisfied by vector functions whose normal component is zero on $\partial\Omega$ (see [7, Exercise II.5.15]), we deduce that the norm $\|\mathbf{S}^{1/2}.\,\|_2$ is also equivalent to $\|\nabla.\,\|_2$ in $\mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$.
- If $\boldsymbol{u} \in \mathbf{W}^{1,2}(\Omega)$ and $\boldsymbol{v} \in \mathbf{H}(\Omega)$ then $(\boldsymbol{u}, \mathbf{T}_2\boldsymbol{v})_2 = (\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_2 = (\mathbf{curl}\,\boldsymbol{u}, \boldsymbol{v})_2$.
- If $\boldsymbol{u} \in \mathbf{W}^{1,2}(\Omega)$ and $\boldsymbol{v} \in D(\mathbf{S})$ then $(\boldsymbol{u}, \mathbf{S}\boldsymbol{v})_2 = (\boldsymbol{u}, \mathbf{T}_2 \circ \mathbf{T}_1\boldsymbol{v})_2 = (\boldsymbol{u}, \mathbf{curl}^2\boldsymbol{v})_2 = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_2$.

Definition 1 *Let $\boldsymbol{u}_0 \in \mathbf{L}^2_\sigma(\Omega)$. A function $\boldsymbol{u} \in L^\infty\big(0,T;\,\mathbf{L}^2_\sigma(\Omega)\big) \cap L^2\big(0,T;\,\mathbf{W}^{1,2}_\sigma(\Omega)\big)$ is said to be a weak solution of the problem (1.1)–(1.4) if*

$$\int_0^T \int_\Omega \big[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\big] \;=\; \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)$$

for all infinitely differentiable functions $\boldsymbol{\phi}$ in $\overline{Q_T}$ such that $\boldsymbol{\phi}(\,.\,,t) \in \mathbf{W}^{1,2}_\sigma(\Omega)$ for all $0 \le t \le T$ and $\boldsymbol{\phi}(\,.\,,T) = \mathbf{0}$.

The boundary condition (1.3b) does not explicitly appear in the weak formulation of the problem (1.1)–(1.4). However, it can be shown by standard methods that if the weak solution is smooth then it satisfies (1.3b).

Definition 2 *Let $\boldsymbol{u}$ be a weak solution of the problem (1.1)–(1.4). We say that $\boldsymbol{u}$ is a strong solution if $\boldsymbol{u}(0) \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$ and $\boldsymbol{u} \in L^r\big(0,T';\, \mathbf{L}^s(\Omega)\big)$ for all $0 < T' \leq T$, $T' < \infty$ and for some $r,\ s \in \mathbb{R}$, satisfying $2 < r < \infty,\ 3 < s < \infty$ and $2/r + 3/s = 1$.*

The following lemma is proved in [10].

Lemma 1 *If $\boldsymbol{u}$ is a strong solution of the problem (1.1)–(1.4) then $\boldsymbol{u} \in L^2\big(0,T';\, \mathbf{W}^{2,2}_{\sigma,nc}(\Omega)\big)$, $\partial_t \boldsymbol{u} \in L^2\big(0,T';\, \mathbf{L}^2_{\sigma}(\Omega)\big)$ for every $0 < T' \leq T$, $T' < \infty$. Moreover, $\boldsymbol{u}$ is a continuous function from $[0,T']$ into $\mathbf{W}^{1,2}_{\sigma}(\Omega)$ (after an eventual change on a set of measure zero).*

2 Local in time existence of strong solutions

In this section we prove the main result of the paper, providing sufficient condition for existence of strong solutions which satisfy the boundary conditions (1.3) on given time interval. Similar results for solutions which satisfy the boundary condition (1.9) were published, e.g., in [5]. J.C. Robinson et al. proved similar results in [14], in which they studied local in time existence of strong solutions with initial data in L^3. They proved sufficient conditions for local in time existence of strong solutions of the Navier-Stokes on the whole space and for local in time existence of strong solutions which satisfy the space-periodic boundary conditions. Our result which is formulated in Theorem 1 is a modification of their result for problem with boundary conditions 1.3 on some smooth bounded convex domain.

Very similar problems were solved in [9], [10], [11] and [12]. In these papers authors studied problems of robustness and stability of strong solutions with respect to perturbations of initial velocities.

In this section we deal with the non-steady Stokes system.

$$\partial_t \boldsymbol{\varphi} + \nu\, \mathbf{curl}^2 \boldsymbol{\varphi} = \mathbf{0} \qquad \text{in } Q_T, \tag{2.1}$$

$$\mathbf{div}\, \boldsymbol{\varphi} = 0 \qquad \text{in } Q_T, \tag{2.2}$$

$$\text{a) } \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0, \quad \text{b) } \mathbf{curl}\, \boldsymbol{\varphi} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on } \partial\Omega \times (0,T), \tag{2.3}$$

$$\boldsymbol{\varphi}(\,.\,,0) = \boldsymbol{\varphi}_0 \qquad \text{in } \Omega. \tag{2.4}$$

Weak solutions of the problem (2.1)–(2.4) are defined the following way:

Definition 3 *Let $\boldsymbol{\varphi}_0 \in \mathbf{L}^2_{\sigma}(\Omega)$. A function $\boldsymbol{\varphi} \in L^{\infty}\big(0,T;\, \mathbf{L}^2_{\sigma}(\Omega)\big) \cap L^2\big(0,T;\, \mathbf{W}^{1,2}_{\sigma}(\Omega)\big)$ is said to be a weak solution of the problem (2.1)–(2.4) if*

$$\int_0^T \int_{\Omega} \big[-\boldsymbol{\varphi} \cdot \partial_t \boldsymbol{\phi} + \nu\, \mathbf{curl}\, \boldsymbol{\varphi} \cdot \mathbf{curl}\, \boldsymbol{\phi}\big] = \int_{\Omega} \boldsymbol{\phi}_0 \cdot \boldsymbol{\varphi}(\,.\,,0)$$

for all infinitely differentiable functions $\boldsymbol{\phi}$ in $\overline{Q_T}$ such that $\boldsymbol{\phi}(\,.\,,t) \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$ for all $0 \leq t \leq T$ and $\boldsymbol{\phi}(\,.\,,T) = \mathbf{0}$.

2.1 The main result

Theorem 1 *There exists an absolute contant* $\epsilon > 0$ *with the following property: Let* $\boldsymbol{u}_0 \in \mathbf{W}^{1,2}_\sigma(\Omega)$, $\tau \in \mathbb{R}$, $0 < \tau \leq T$ *and* $\boldsymbol{\varphi}$ *be a solution of (2.1)–(2.4) with initial velocity* $\boldsymbol{u}_0$ *such that*

$$\|\boldsymbol{u}_0\|_3 \int_0^\tau \int_\Omega |\nabla \boldsymbol{\varphi}|^2 |\boldsymbol{\varphi}| < \epsilon. \tag{2.5}$$

Then there exists a strong solution $\boldsymbol{u}$ *of the problem (1.1)–(1.4) on the time interval* $(0, \tau)$.

2.2 The mathematical model of behaviour of fluid

We study qualitative properties of solutions of the Navier-Stokes equations with the boundary conditions of the Navier's type in this contribution. We derive a criterion for local in time existence of strong solution. Although this result looks to be a theoretical one, it has a very specific relationship to fluid dynamics. The system of the Navier-Stokes equations which is solved in this contribution is a model of behaviour of incompressible fluid in bounded domains. Equations (1.1) which are called the Navier-Stokes equations are derived from the momentum conservation law. Equation (2.2) which is called the continuity equation expresses the fact that liquid is incompressible. Conditions (1.3) describe interaction between fluid and fixed wall. We are supposing density and viscosity of fluid are constant, in this model.

We can say that the system (1.1), (1.2), (1.4) with various types of boundary conditions (we are studying this system with boundary conditions (1.3) here) is the most commonly used model of behaviour of incompressible fluid. This model is studied in at least two areas of mathematics. Many mathematicians dealing with the theory of partial differential equations publish papers in which the qualitative properties of solutions of this model are studied. At the same time, many mathematicians working in the field of numerical mathematics publish papers that deal with the approximate numerical solutions of this model. These approximate solutions are then applied to solve various technical problems in different areas, mechanical engineering, civil engineering, hydrodynamics. It should be said that the results dealing with the qualitative properties of this system often help to optimize work of numerical mathematicians.

Unfortunately, mathematicians and physicists are not yet sure if this mathematical model corresponds to the real fluid behavior. In the qualitative theory of the Navier-Stokes equations there are still some open problems that are of crucial importance. Suppose we have a weak solution to the problem with a smooth enough initial velocity. It is well known that there is a weak solution of this problem, satisfying the energy inequality in addition. This means that its kinetic energy is uniformly bounded over the entire time interval. We do not know whether this solution satisfies energy identity (energy identity is a special case of energy inequality). This means we do not know whether or not the fluid behavior corresponds with the energy conservation law. The next open question remains whether or not the dissipative energy is uniformly limited over the entire time interval. If we knew that there exists a weak solution of the problem which is also a strong solution, all of the above mentioned questions would be positively answered. Such a solution would satisfy energy identity and its dissipative energy would be uniformly bounded over the time interval. We also know that if there exists a strong solution, then this solution is the unique strong solution. In addition, it is the only solution that satisfies energy equality. It is even known that such a solution would be then the only solution that satisfies the energy inequality. It is therefore reasonable to think that such a solution would be a mathematical model that actually describes the behavior of the fluid.

The open problem mentioned here is one of the most well-known open problems of applied mathematics, included in the so-called millennium problems, and the Clay Institute has issued a financial reward for solution of this problem.

The main result of this paper (Theorem 1) provides a partial contribution to the solution of this problem. We know that there is a time interval on which the solution is strong, and we know how to determine the length of this interval. This means that we know that at this time interval the solution of the Navier-Stokes problem describes the behavior of a fluid that complies with the law of conservation of energy.The question of whether or not this solution is strong outside this time interval remains open.

2.3 Proof of the main result

At first we mention some estimates and lemmas which we apply in order to prove Theorem 1. The following estimate is proved in [11, Estimate (2.1)]. If $\boldsymbol{v} \in \mathbf{W}^{1,3}(\Omega)$, $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial(\Omega)$, then

$$\|\boldsymbol{v}\|_9^3 \leq c_3 \int_\Omega |\nabla \boldsymbol{v}|^2 |\boldsymbol{v}|. \tag{2.6}$$

Let $\boldsymbol{v} \in \mathbf{W}^{1,2}_\sigma(\Omega)$. Then there exists $q \in \mathbf{W}^{1,2}(\Omega)$ such that

$$\boldsymbol{v}|\boldsymbol{v}| = \mathbf{P}_\sigma(\boldsymbol{v}|\boldsymbol{v}|) + \nabla q. \tag{2.7}$$

Applying operator $\mathbf{div}$ to 2.7 and using the fact that $\boldsymbol{v} \cdot \boldsymbol{n} \equiv 0$ on $\partial\Omega$ we obtain

$$\Delta q = \mathbf{div}\,(\boldsymbol{v}|\boldsymbol{v}|) \quad \text{in } \Omega \qquad \frac{\partial q}{\partial \boldsymbol{n}} = 0 \quad \text{in } \partial\Omega \tag{2.8}$$

Therefore ,

$$\left\|D^2 q\right\|_{3/2} \leq c_4 \left\| \nabla \boldsymbol{v}|\boldsymbol{v}| \right\|_{3/2} \leq c_5 \left\|\nabla \boldsymbol{v}|\boldsymbol{v}|\right\|_2 \|\boldsymbol{v}\|_3^{1/2}. \tag{2.9}$$

The following inequalities are proved in [10, Lemma 2.] where assumption of convexity of Ω has been applied. Let $\boldsymbol{w} \in \mathbf{W}^{2,2}_{\sigma,nc}(\Omega)$. Then

$$\int_\Omega \mathbf{S} \cdot \mathbf{P}_\sigma(\boldsymbol{v}|\boldsymbol{v}|) \equiv -\int_\Omega \Delta \boldsymbol{v} \cdot \mathbf{P}_\sigma(\boldsymbol{v}|\boldsymbol{v}|) \geq \int_\Omega |\nabla \boldsymbol{v}|^2 |\boldsymbol{v}| + \frac{4}{9} \int_\Omega \left|\nabla |\boldsymbol{v}|^{3/2}\right|^2. \tag{2.10}$$

The following lemma is proved in [14].

Lemma 2 *Let $c_6 > 0$, θ,ϑ and ς are real-valued, non-negative functions which are continuous on $[0, \tau)$, $\theta \in C(0, \tau)$, $\theta(0) = 0$ and the inequality*

$$\theta'(t) + \vartheta(t) \leq c_6 \theta(t)\vartheta(t) + \varsigma(t) \tag{2.11}$$

holds on $(0, \tau)$. Put $D = \int_0^\tau \varsigma(t)\, dt$. If $D < \frac{1}{4c_6}$ then

$$\sup_{t \in (0,\tau)} \theta(t) \leq 2D < \frac{1}{2c_6} \tag{2.12}$$

and

$$\int_0^\tau \vartheta(t)\, dt \leq 2D < \frac{1}{2c_6}. \tag{2.13}$$

Integrating (2.1) and using (2.10) and embedding $\mathbf{W}^{1,2}_{\sigma}(\Omega) \hookrightarrow \mathbf{L}^3(\Omega)$ we prove the following lemma.

Lemma 3 *Let $\varphi_0 \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$, $\boldsymbol{\varphi}$ be a solution of (2.1)–(2.4). Then the inequality*

$$\|\boldsymbol{\varphi}(T')\|_3^3 + \int_0^{T'} \int_{\Omega} |\nabla \boldsymbol{\varphi}|^2 |\boldsymbol{\varphi}| \leq \|\boldsymbol{\varphi}_0\|_3^3 \tag{2.14}$$

holds for every T', $0 < T' \leq T$.

Now we prove Theorem 1:

It is well known that if $\boldsymbol{u}_0 \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$ then $\boldsymbol{\varphi} \in L^2\big(0, T'; \mathbf{W}^{2,2}_{\sigma,nc}(\Omega)\big)$. Multiplying (2.1) by $\mathbf{P}_{\sigma}(\varphi|\varphi|)$, integrating it over Ω and $(0, T')$ and using (2.10) we obtain (2.14) with $\boldsymbol{u}_0$ instead of $\boldsymbol{\varphi}_0$. Let

$$\varsigma = \sup\{\varsigma';\ \varsigma',\ \text{such that } \boldsymbol{u} \text{ is a strong solution on } (0,\varsigma')\}. \tag{2.15}$$

Since $\boldsymbol{u}_0 \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$ then $\varsigma > 0$. This statement can be proven by analogy with [8, Theorem 6.1]. We prove that $\varsigma \geq \tau$. Suppose by contradiction that $\varsigma < \tau$. It is sufficient to prove that

$$\boldsymbol{u} \in C([0,\varsigma];\ \mathbf{W}^{1,2}_{\sigma}(\Omega)). \tag{2.16}$$

By (2.16) $\boldsymbol{u}(\varsigma) \in \mathbf{W}^{1,2}_{\sigma}(\Omega)$. Consequently, there exists $\varsigma^* > \varsigma$ such that $\boldsymbol{u}$ is a strong solution on $(0, \varsigma^*)$ and we obtain the contradiction with (2.15).

Let $\boldsymbol{u} = \boldsymbol{\varphi} + \boldsymbol{\psi}$ such that $\boldsymbol{\psi}$ is solution of the system

$$\boldsymbol{\psi}' - \Delta\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\varphi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi} + \nabla P = 0, \tag{2.17}$$

$$\boldsymbol{\psi}(0) = 0 \tag{2.18}$$

and $\boldsymbol{\varphi}$ is solution of the problem (2.1)–(2.4) with initial velocity $\boldsymbol{u}_0$. By (1.6) there exists $q \in \mathbf{W}^{1,3/2}(\Omega)$ such that

$$\boldsymbol{\psi}|\boldsymbol{\psi}| = \mathbf{P}_{\sigma}(\boldsymbol{\psi}|\boldsymbol{\psi}|) + \nabla q. \tag{2.19}$$

Multiplying (2.17) by $\mathbf{P}_{\sigma}(\boldsymbol{\psi}|\boldsymbol{\psi}|)$ on the time interval $(0,\varsigma)$, integrating it over Ω and using (2.7), (2.10), (2.19) and a well-known identity (see, e.g., [15, Lemma 3.2.1])

$$\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi}\,,\ \boldsymbol{\psi}|\boldsymbol{\psi}|\big)\!\big) = \big(\!\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\psi}\,,\ \boldsymbol{\psi}|\boldsymbol{\psi}|\big)\!\big) = 0$$

we obtain

$$\begin{aligned}
\frac{1}{3}\frac{d}{dt}\|\boldsymbol{\psi}\|_3 &+ \|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|^{1/2}\|_2^2 + \frac{4}{9}\|\nabla|\boldsymbol{\psi}|^{3/2}\|_2^2 \leq \\
&\big|\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\varphi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \boldsymbol{\psi}|\boldsymbol{\psi}|\big)\!\big)\big| + \\
&\big|\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\psi} + (\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\varphi} + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \nabla q\big)\!\big)\big| \leq \\
&\big|\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \boldsymbol{\psi}|\boldsymbol{\psi}|\big)\!\big)\big| + \big|\big(\!\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \boldsymbol{\psi}|\boldsymbol{\psi}|\big)\!\big)\big| + \big|\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\psi}\,,\ \nabla q\big)\!\big)\big| + \\
&\big|\big(\!\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \nabla q\big)\!\big)\big| + \big|\big(\!\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\psi}\,,\ \nabla q\big)\!\big)\big| + \big|\big(\!\big((\boldsymbol{\psi}\cdot\nabla)\boldsymbol{\varphi}\,,\ \nabla q\big)\!\big)\big| = \\
&I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\end{aligned} \tag{2.20}$$

Integrating by parts and applying (2.6) and (2.9) we infer

$$I_1 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_{3/2}\|\boldsymbol{\psi}\|_6\|\boldsymbol{\varphi}\|_6 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2\|\boldsymbol{\psi}\|_3^{1/2}\|\boldsymbol{\psi}\|_9^{3/4}\|\boldsymbol{\psi}\|_3^{1/4}\|\boldsymbol{\varphi}\|_9^{3/4}\|\boldsymbol{\varphi}\|_3^{1/4} \le$$
$$c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^{3/2}\|\boldsymbol{\psi}\|_3^{3/4}\|\boldsymbol{\varphi}\|_9^{3/4}\|\boldsymbol{\varphi}\|_3^{1/4} \le c_7\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3^3 + \frac{1}{2}\big\|\nabla\boldsymbol{\varphi}|\boldsymbol{\varphi}|\big\|_2^2\|\boldsymbol{u}_0\|_3 \qquad (2.21)$$

$$I_2 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_{3/2}\|\boldsymbol{\varphi}\|_6^2 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2\|\boldsymbol{\psi}\|_3^{1/2}\|\boldsymbol{\varphi}\|_9^{3/2}\|\boldsymbol{\varphi}\|_3^{1/2} \le$$
$$\frac{1}{8}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2 + c_8\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3^3 + \frac{1}{2}\big\|\nabla\boldsymbol{\varphi}|\boldsymbol{\varphi}|\big\|_2^2\|\boldsymbol{u}_0\|_3 \qquad (2.22)$$

$$I_3 \le c\,\big\|D^2 q\big\|_{3/2}\|\boldsymbol{\psi}\|_6^2 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_{3/2}\|\boldsymbol{\psi}\|_9^{3/2}\|\boldsymbol{\psi}\|_3^{1/2} \le$$
$$c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3 \le \frac{1}{8}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2 + c_9\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3^3 \qquad (2.23)$$

$$I_4 \le c\,\big\|D^2 q\big\|_{3/2}\|\boldsymbol{\varphi}\|_6^2 \le c\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_{3/2}\|\boldsymbol{\varphi}\|_9^{3/2}\|\boldsymbol{\varphi}\|_3^{1/2} \le$$
$$\frac{1}{8}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2 + c_{10}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3^3 + \big\|\nabla\boldsymbol{\varphi}|\boldsymbol{\varphi}|\big\|_2^2\|\boldsymbol{u}_0\|_3 \qquad (2.24)$$

$$I_5 + I_6 \le c\,\big(I_3 + I_4\big) \le \frac{1}{8}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2 + c_{11}\,\big\|\nabla\boldsymbol{\psi}|\boldsymbol{\psi}|\big\|_2^2\|\boldsymbol{\psi}\|_3^3 + \big\|\nabla\boldsymbol{\varphi}|\boldsymbol{\varphi}|\big\|_2^2\|\boldsymbol{u}_0\|_3 \qquad (2.25)$$

Set $c_{12} = c_7 + c_8 + c_9 + c_{10} + c_{11}$. Using estimates (2.20)–(2.25) we obtain

$$\frac{d}{dt}\|\boldsymbol{\psi}\|_3^3 + \big\|\nabla\boldsymbol{\psi}\,|\boldsymbol{\psi}|^{1/2}\big\|_2^2 \le 3\,c_{12}\,\big\|\nabla\boldsymbol{\psi}\,|\boldsymbol{\psi}|^{1/2}\big\|_2^2\,\|\boldsymbol{\psi}\|_3^3 + 3\,\|\boldsymbol{u}_0\|_3\big\|\nabla\boldsymbol{\varphi}\,|\boldsymbol{\varphi}|^{1/2}\big\|_2^2.$$

Put $\theta(t) = \|\boldsymbol{\psi}(.,t)\|_3^3$, $\vartheta(t) = \big\|\nabla\boldsymbol{\psi}(.,t)|\boldsymbol{\psi}|^{1/2}\big\|_2^2$, $\varsigma(t) = 3\,\|\boldsymbol{u}_0\|_3\big\|\nabla\boldsymbol{\varphi}\,|\boldsymbol{\varphi}|^{1/2}\big\|_2^2$ and $\epsilon = \frac{1}{12c_{12}}$. Let $0 < \tau < T$ such that

$$\|\boldsymbol{u}_0\|_3 \int_0^\tau \big\|\nabla\boldsymbol{\varphi}\,|\boldsymbol{\varphi}|^{1/2}\big\|_2^2 < \frac{1}{12c_{12}}.$$

If $\tau \le \varsigma$ the theorem is proved. Suppose by contradiction $\varsigma < \tau$. Since

$$3\,\|\boldsymbol{u}_0\|_3 \int_0^\varsigma \big\|\nabla\boldsymbol{\varphi}\,|\boldsymbol{\varphi}|^{1/2}\big\|_2^2 < \frac{1}{4c_{12}}.$$

we get by Lemma 2 and by (2.6)

$$\int_0^\varsigma \|\boldsymbol{\psi}\|_9^3 \le c_3 \int_0^\varsigma \big\|\nabla\boldsymbol{\psi}\,|\boldsymbol{\psi}|^{1/2}\big\|_2^2 < \frac{1}{2c_{12}} < \infty.$$

The last inequalities, (2.6) and (2.14) imply that $\boldsymbol{u} \in L^3(0,\varsigma;\,\mathbf{L}^9(\Omega))$. Then, $\boldsymbol{u} \in C([0,\varsigma];\,\mathbf{W}^{1,2}_\sigma(\Omega))$. Since (2.16) holds we obtain the contradiction with (2.15). Consequently, $\tau \le \varsigma$ and theorem is proved.

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