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LIENHARD INTERPOLATION METHOD, ITS GENERALISATION AND ADDITION

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Abstract. This article is based on the primary publication LIENHARD, H.: Interpolation von Funktionswerten bei numerischen Bahnsteuerungen, furthermore from the publication MATUŠŮ, J. and MATUŠŮ, M.: The Lienhard Interpolation $L_{Q,p}$ – method (see the list of literature), in which the original structure of interpolation was extended. This presented article follows directly on $L_{Q,p}$ – method, which expands $L_{Q,p}$ – method in two ways. The first way relates to the enumeration of elements on the basic matrix of $L_{Q,p}$ – method, with its assistance it is possible to obtain parametric equations of interpolation curve. The other way contains parametric equations of interpolation curve in a situation in which the parameter for each part of the curve is changing at the given interval limited by the extreme (node) points.

Keywords: Lienhard interpolation method, elements of the matrix calculation, system of linear equation and its solution

Mathematics Subject Classification: Primary 65D05, 65D07, 65D15;
Secondary 65D10, 65D17, 41-00, 41-01, 68U05

1 Introduction

Let $Q \geq 1, n \geq 3$ be integers. In the Euclidean space $R^m (m > 1 \text{ integer})$, we think of n different points $P_i = x_j^{(i)} (i = 1, \dots, n;\ j = 1, \dots, m)$. We search for the polynomials of real variables t of the highest K-th degree

$$P_j^{(i)}(t) = \sum_{k=0}^{K} a_{jk}^{(i)} t^k \tag{1}$$

for $i = 1, \dots, n-1$ (the case of the opened interpolation curve), or for $i = 1, \dots, n$ (the case of closed interpolation curve, where for the point $P_{n+1} = x_j^{(n+1)}$, we understand the point $P_1 = x_j^{(1)}$), and the way that applies

$$P_j^{(i)}(-1) = x_j^{(i)}, \qquad P_j^{(i)}(1) = x_j^{(i+1)} \tag{2}$$

and furthermore $(q = 1, \dots, Q)$

$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} P_j^{(i)}(1) = \frac{\mathrm{d}^q}{\mathrm{d}t^q} P_j^{(i+1)}(-1) \tag{3}$$

for $i = 1, \dots, n-2$ (the opened interpolation curve), or for $i = 1, \dots, n$ (closed interpolation curve, where for the function $P_j^{(n+1)}(t)$, we understand the function $P_j^{(1)}(t)$).

Conditions (2) guarantee, that the i-th segment of the curve parameterised by the function $P_j^{(i)}(t)$ $(j = 1, \ldots, m)$, goes through points P_i, P_{i+1}; based on conditions (3) the transition from the i-th segment to $(i+1)$-st segment implements smoothly from 1st to Q-th derivation. To be brief we add

$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} P_j^{(i)}(-1) = \mathrm{D}^q x_j^{(i)}, \; \frac{\mathrm{d}^q}{\mathrm{d}t^q} P_j^{(i)}(1) = \mathrm{D}^q x_j^{(i+1)} \quad . \tag{4}$$

We will talk later about the way to define values $\mathrm{D}^q x_j^{(i)}, \mathrm{D}^q x_j^{(i+1)}$.

By (2), (3) we have $2Q + 2$ conditions available, from their bases is each polynomial (1) of the highest degree $K = 2Q + 1$:

$$P_j^{(i)}(t) = \sum_{k=0}^{2Q+1} a_{jk}^{(i)} t^k \quad . \tag{5}$$

For q-th derivation applies

$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} P_j^{(i)}(t) = \sum_{k=0}^{2Q+1} k(k-1)\ldots(k-q+1) a_{jk}^{(i)} t^{k-q} = \tag{6}$$

$$= \sum_{k=0}^{2Q+1} q! \binom{k}{q} a_{jk}^{(i)} t^{k-q} \quad .$$

After setting values $t = -1{,}1$ to (5), (6), we get (while taking into account (2), (4)) the following system of $2Q + 2$ linear equation for $2Q + 2$ undefined coefficients $a_{jk}^{(i)}$:

$$\sum_{k=0}^{2Q+1} (-1)^k a_{jk}^{(i)} = x_j^{(i)} \quad , \tag{7}$$

$$\sum_{k=0}^{2Q+1} (-1)^{k-q} q! \binom{k}{q} a_{jk}^{(i)} = \mathrm{D}^q x_j^{(i)} \quad ,$$

$$(q = 1, \ldots, Q) \; .$$

$$\sum_{k=0}^{2Q+1} a_{jk}^{(i)} = x_j^{(i+1)} \; ,$$

$$\sum_{k=0}^{2Q+1} q! \binom{k}{q} a_{jk}^{(i)} = \mathrm{D}^q x_j^{(i+1)} \quad ,$$

$$(q = 1, \ldots, Q) \; .$$

We implement matrices

$$A_{ij} = (a_{j0}^{(i)}, a_{j1}^{(i)}, \ldots, a_{j,2Q+1}^{(i)}) \; , \tag{8}$$

$$X_{ij} = (x_j^{(i)}, X_{ij}^+, x_j^{(i+1)}, X_{i+1,j}^+) \; , \tag{9}$$

where

$$X_{ij}^+ = (\mathrm{D}^1 x_j^{(i)}, \mathrm{D}^2 x_j^{(i)}, \ldots, \mathrm{D}^Q x_j^{(i)}) \tag{10}$$

$$X_{i+1,j}^+ = (\mathrm{D}^1 x_j^{(i+1)}, \mathrm{D}^2 x_j^{(i+1)}, \ldots, \mathrm{D}^Q x_j^{(i+1)}) \tag{11}$$

and let A_Q be the system matrix (7). We call it the basic matrix of method $L_{Q,p}$. If A_Q is regular, then the solution of the system (7): $A_Q \cdot A_{ij}^T = X_{ij}^T$, equals to

$$A_{ij}^T = A_Q^{-1} \cdot X_{ij}^T \,, \tag{12}$$

where A_{ij}^T, X_{ij}^T are transposed matrices to matrices (8), (9) and A_Q^{-1} is the inverse matrix of the matrix A_Q. Due to (12), polynoms (5) are:

$$\begin{aligned} P_j^{(i)}(t) &= (1, t, t^2, \dots, t^{2Q+1}) \cdot A_{ij}^T = \\ &= (1, t, t^2, \dots, t^{2Q+1}) \cdot (A_Q^{-1} \cdot X_{ij}^T) \,. \end{aligned} \tag{13}$$

2 Grouping of Support Points

Let's start with the case of the opened interpolation curve $P_1 P_2 \dots P_n$. Let $W = 2n - 2$ and to the final sequence of support points $P_1 P_2 \dots P_n$, we add these points to the right

$$P_{n+1} = P_{n-1}, P_{n+2} = P_{n-2}, \dots, P_W = P_{2n-W} \qquad . \tag{14}$$

For the given integer k, we determine the smallest non-negative remainder r when dividing the number k by the number W and set

$$P_k = \begin{cases} P_r & for\ r > 0 \,, \\ P_W & for\ r = 0 \,. \end{cases} \tag{15}$$

Therefore, we get the infinite sequence of points

$$\dots, P_{-5}, P_{-4}, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, P_4, P_5, \dots \tag{16}$$

For instance, for $n = 5$ we think of an opened interpolation curve $P_1 P_2 P_3 P_4 P_5$. Set $W = 8$ and to the sequence of points P_1, P_2, P_3, P_4, P_5 we add these points to the right $P_6 = P_4, P_7 = P_3, P_8 = P_2$. For instance, for $k = 9$ applies $9 = 1 \cdot 8 + 1$, and therefore $P_9 = P_1$, or for $k = 10$ applies $10 = 1 \cdot 8 + 2$, and therefore $P_{10} = P_2$.
Then $P_{11} = P_3, P_{12} = P_4, P_{13} = P_5, P_1 = P_6 = P_4, P_{15} = P_7 = P_3$. For $k = 0$ applies $0 = 0 \cdot 8 + 0$, therefore $P_0 = P_8 = P_2$, for $k = -1$ applies $-1 = (-1) \cdot 8 + 7$, therefore $P_{-1} = P_7 = P_3$. For instance, for $k = -315$ applies $-315 = (-40) \cdot 8 + 5$, therefore $P_{-315} = P_5$. The sequence (16) will then be composed of points

$$\dots, P_{-2} = P_{-4}, P_{-1} = P_3, P_0 = P_2,\ P_1, P_2, P_3, P_4, P_5, P_6 = P_4, P_7 = P_3, P_8 = P_2 \dots \tag{17}$$

In case of the closed interpolation curve $P_1 P_2 \dots P_n P_1$, we will proceed as follows. Set $W = n$. For the given integer k, we determine the smallest non-negative remainder r when dividing the number k by the number W, and set

$$P_k = \begin{cases} P_r & for\ r > 0 \,, \\ P_W & for\ r = 0 \,. \end{cases} \tag{18}$$

Thus, once again, we get the infinite sequence of points (16). For instance, $n = 4$ will be the closed interpolation curve $P_1 P_2 P_3 P_4 P_1$. For $k = 5$ applies $5 = 1 \cdot 4 + 1$, therefore $P_5 = P_1$, then $P_6 = P_2, P_7 = P_3, P_8 = P_4$. Furthermore $P_0 = P_4, P_{-1} = P_3, P_{-2} = P_2, P_{-3} = P_1, P_{-4} = P_4$. For instance, for $k = -1026$ applies $-1026 = (-257) \cdot 4 + 2$, therefore $P_{-1026} = P_2$. The sequence (16) is then composed of points

$$\dots, P_4, P_1, P_2, P_3, P_4, P_1, P_2, P_3, P_4, P_1, P_2, P_3, P_4, \dots \tag{19}$$

3 The Definition of Derivative Values of Parametric Functions

We now define values from 1st to Q-th derivation of function $P_j^{(i)}(t)$ at points $t = -1,1$. For that purpose, we display on the plane t, x_j points $(2h, x_j^{(i+h)})$, where $-Q + p \le h \le Q - p$, h integer; with the given Q representing p of any number fulfilling the inequality $0 \le p \le Q - 1$. The value $x_j^{(i+h)}$, we understand as the j-th coordinate of the point P_{i+h} in the sequence (16) while respecting the type of interpolation curve (opened or closed), see the figure 1.

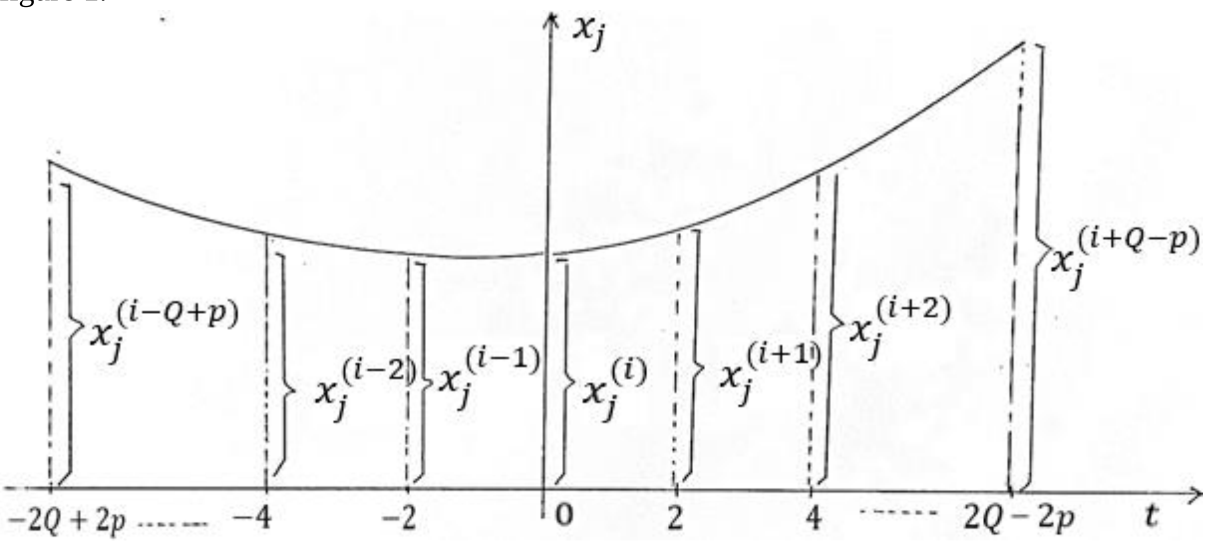

Fig. 1

Through these points, with their quantity equalled to the number $2Q - 2p + 1$, the polynomial of degree $2Q - 2p$ is then unambiguously determined

$$R_j^{(i)}(t) = \sum_{k=0}^{2Q-2p} b_{jk}^{(i)} t^k \, . \tag{20}$$

And just by using the polynomial (20), we define the derivation $\mathrm{D}^q x_j^{(i)}$ with relation

$$\mathrm{D}^q x_j^{(i)} = \frac{\mathrm{d}^q}{\mathrm{d}t^q} R_j^{(i)}(0) \tag{21}$$

$$(q = 1, \dots, Q).$$

Then we can easily find out that for $2Q - 2p \ge Q$, i.e. for $Q \ge 2p$, the following applies

$$\mathrm{D}^q x_j^{(i)} = q!\, b_{jq}^{(i)} \tag{22}$$

$$(q = 1, \dots, Q),$$

for $2Q - 2p < Q$, i.e. for $Q < 2p$, the following applies

$$\mathrm{D}^q x_j^{(i)} = q!\, b_{jq}^{(i)} \tag{23}$$

$$(q = 1, \dots, 2Q - 2p),$$

and

$$\mathrm{D}^q x_j^{(i)} = 0 \tag{24}$$

$$(q = 2Q - 2p + 1, \dots, Q).$$

3.1 Note

The derivative values of parametric functions $P_j^{(i)}(t)$ at points $t = -1{,}1$ is then equalled to the derivative values of the auxiliary polynomial (20) at the point zero. The meaning of the integer $p \in < 0, Q - 1 >$ is that for making these derivations, we more or less need given support points; for $p = 0$ the quantity is maximum $(= 2Q + 1)$, for $p = Q - 1$ minimum $(= 3)$. With the given Q, the number p affects the shape of the resulting interpolation curve. For instance, if $Q = 3, p = 2$ then, according to (24), $\mathrm{D}^3 x_j^{(i)} = 0$.

4 The Universal Matrix of Method $L_{Q,p}$

Because each coefficient of polynomial (20) is a linear combination of values $x_j^{(i-Q+p)}, x_j^{(i-Q+p+1)}, \dots, x_j^{(i+Q-p)}$, also every derivation $\mathrm{D}^q x_j^{(i)}$ $(q = 1, \dots, Q)$ consists of some linear combination of those values. Therefore, there also exists a numerical matrix $B_{Q,p}$ of type $(Q, 2Q - 2p + 1)$ that is (see (10))

$$X_{ij}^{+} = \left(x_j^{(i-Q+p)}, x_j^{(i-Q+p+1)}, \dots, x_j^{(i+Q-p)}\right) \cdot B_{Q,p}^{T}, \tag{25}$$

and similarly (see (11)),

$$X_{i+1,p}^{+} = \left(x_j^{(i-Q+p+1)}, x_j^{(i-Q+p+2)}, \dots, x_j^{(i+Q-p+1)}\right). \tag{26}$$

By continuing in a similar way, to the one described in a publication [4], we get a matrix (8), described in a following way:

$$A_{ij}^{T} = C_{Q,p} \cdot \begin{pmatrix} x_j^{(i-Q+p)} \\ x_j^{(i-Q+p+1)} \\ \vdots \\ x_j^{(i+Q-p+1)} \end{pmatrix} \quad ,$$

where

$$C_{Q,p} = A_Q^{-1} \cdot \tag{27}$$

$$\begin{array}{r c}
 & \begin{array}{cccccccc} & & & Q\text{-}p\text{+}1 & Q\text{-}p\text{+}2 & & & \end{array} \\
\begin{array}{r} \\ \\ \\ \\ \cdot\, Q+2 \\ \\ \\ \\ \end{array} & \left(\begin{array}{cccccccc}
0 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
 & & & & & & & 0 \\
 & & & \boxed{B_{Q,p}} & & & & \vdots \\
 & & & & & & & 0 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 0 \\
0 & & & & & & & \\
\vdots & & & & \boxed{B_{Q,p}} & & & \\
0 & & & & & & &
\end{array}\right)
\end{array}$$

is the matrix of type $(2Q + 2, 2Q - 2p + 2)$. We call it the universal matrix of method $L_{Q,p}$. The final form of interpolation polynomials is then

$$P_j^{(i)}(t) = (1, t, t^2, \dots, t^{2Q+1}) \cdot C_{Q,p} \cdot \begin{pmatrix} x_j^{(i-Q+p)} \\ x_j^{(i-Q+p+1)} \\ \vdots \\ \vdots \\ x_j^{(i+Q-p+1)} \end{pmatrix}. \tag{28}$$

5 The Calculatuion of Basic Matrix Elements of Method $L_{Q,p}$

If we mark elements of the matrix A_Q system of equation (7) with the symbol m_{rs}, then $A_Q = (m_{rs})_{1 \le r,s \le 2Q+2}$, the matrix A_Q is of the type $(2Q + 2, 2Q + 2)$.
Applies

$$m_{rs} = \tag{29}$$

$$= \begin{cases} (-1)^{s-1} & \text{for } r = 1 \text{ and } 1 \le s \le 2Q + 2 \\ 0 & \text{for } 2 \le r \le Q + 1 \text{ and } 1 \le s \le r - 1 \\ (-1)^{s-1}(r-1)!\binom{s-1}{r-1} & \text{for } 2 \le r \le Q + 1 \text{ and } r \le s \le 2Q + 2 \\ 1 & \text{for } r = Q + 2 \text{ and } 1 \le s \le 2Q + 2 \\ \left|m_{r-(Q+1),s}\right| & \text{for } Q + 3 \le r \le 2Q + 2 \text{ and } 1 \le s \le 2Q + 2 \end{cases}.$$

According to (32), we find out that e.g. the element m_{43} of the matrix A_1 is equal to

$$m_{43} = |m_{23}| = (-1)^1 1! \binom{2}{1} = 2\,,$$

the element $m_{33} = 1$ and

$$m_{24} = (-1)^2 1! \binom{3}{1} = 3 \, .$$

The whole matrix A_1 is

$$A_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \tag{30}$$

and the corresponding inverse matrix is

$$A_1^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix} . \tag{31}$$

According to (29), we find out that e.g. the element m_{26} of the matrix A_2 equals to

$$m_{26} = (-1)^4 1! \binom{5}{1} = 5 \, ,$$

the element $m_{32} = 0$ and

$$m_{54} = |m_{24}| = \left|(-1)^2 1! \binom{3}{1}\right| = 3 \, .$$

The whole matrix A_2 is

$$A_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 2 & -6 & 12 & -20 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{pmatrix} \tag{32}$$

and the corresponding inverse matrix is

$$A_2^{-1} = \frac{1}{16} \begin{pmatrix} 8 & 5 & 1 & 8 & -5 & 1 \\ -15 & -7 & -1 & 15 & -7 & 1 \\ 0 & -6 & -2 & 0 & 6 & -2 \\ 10 & 10 & 2 & -10 & 10 & -2 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ -3 & -3 & - & 3 & -3 & 1 \end{pmatrix} . \tag{33}$$

According to (29), we find out that e.g. the element m_{37} of the matrix A_3 equals to

$$m_{37} = (-1)^4 2! \binom{6}{2} = 30 \, ,$$

the element $m_{83} = 0$ and

$$m_{88} = |m_{48}| = \left|(-1)^4 3! \binom{7}{3}\right| = 210 \, .$$

The whole matrix A_3 is

$$A_3 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 & -6 & 7 \\ 0 & 0 & 2 & -6 & 12 & -20 & 30 & -42 \\ 0 & 0 & 0 & 6 & -24 & 60 & -120 & 210 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 & 42 \\ 0 & 0 & 0 & 6 & 24 & 60 & 120 & 210 \end{pmatrix} \tag{34}$$

and the corresponding inverse matrix is

$$A_3^{-1} = \frac{1}{96} \begin{pmatrix} 48 & 33 & 9 & 1 & 48 & -33 & 9 & -1 \\ -105 & -57 & -12 & -1 & 105 & -57 & 12 & -1 \\ 0 & -45 & -21 & -3 & 0 & 45 & -21 & 3 \\ 105 & 105 & 30 & 3 & -105 & 105 & -30 & 3 \\ 0 & 15 & 15 & 3 & 0 & -15 & 15 & -3 \\ -63 & -63 & -24 & -3 & 63 & -63 & 24 & -3 \\ 0 & -3 & -3 & -1 & 0 & 3 & -3 & 1 \\ 15 & 15 & 6 & 1 & -15 & 15 & -6 & 1 \end{pmatrix} . \tag{35}$$

For interests, e.g. the element $m_{16,15}$ of the matrix A_9 which is the type (20,20), according to (29) equals

$$m_{16,15} = |m_{6,15}| = \left|(-1)^9 5! \binom{14}{5}\right| = 240240 \, .$$

We won't show the whole matrix A_9 and the corresponding inverse matrix A_9^{-1} either but they do exist.

5.1 Example

On the plane R^2, we think of support points $P_1 = (1,1), P_2 = (2,3), P_3 = (5,-1), P_4 = (2,-3), P_5 = (4,5)$. With the method $L_{3,1}$, we create an opened interpolation curve $P_1P_2P_3P_4P_5$. The matrix system of equations (7) is A_3 (which is the matrix (34)) with the according inverse matrix (35). For the purpose of getting the matrix $B_{3,1}$ of the type (3,5) (see (25)), we add to the polynomial (see (20))

$$R_j^{(i)}(t) = \sum_{k=0}^{4} b_{jk}^{(i)} t^k \tag{36}$$

points $(2h, x_j^{(i+h)}), -2 \le h \le 2, h$ integer. Polynomial coefficients (36) are calculated with a simple calculation

$$b_{j0}^{(i)} = x_j^{(i)} ,$$

$$b_{j1}^{(i)} = (-x_j^{(i+2)} + 8x_j^{(i+1)} - 8x_j^{(i-1)} + x_j^{(i-2)})/24 ,$$

$$b_{j2}^{(i)} = (-x_j^{(i+2)} + 16x_j^{(i+1)} - 30x_j^{(i)} + 16x_j^{(i-1)} - x_j^{(i-2)})/96 ,$$

$$b_{j3}^{(i)} = (x_j^{(i+2)} - 2x_j^{(i+1)} + 2x_j^{(i-1)} - x_j^{(i-2)})/96 ,$$

$$b_{j4}^{(i)} = (x_j^{(i+2)} - 4x_j^{(i+1)} + 6x_j^{(i)} - 4x_j^{(i-1)} + x_j^{(i-2)})/384 .$$

According to (22) which is $\mathrm{D}^1 x_j^{(i)} = b_{j1}^{(i)}, \mathrm{D}^2 x_j^{(i)} = 2b_{j2}^{(i)}, \mathrm{D}^3 x_j^{(i)} = 6b_{j3}^{(i)}$, so (see (25))

$$X_{ij}^{+} = \left(\mathrm{D}^1 x_j^{(i)}, \mathrm{D}^2 x_j^{(i)}, \mathrm{D}^3 x_j^{(i)}\right) =,$$

$$= (x_j^{(i-2)}, x_j^{(i-1)}, x_j^{(i)}, x_j^{(i+1)}, x_j^{(i+2)}) \cdot B_{3,1}^T ,$$

where

$$B_{3,1} = \frac{1}{48}\begin{pmatrix} 2 & -16 & 0 & 16 & -2 \\ -1 & 16 & -30 & 16 & -1 \\ -3 & 6 & 0 & -6 & 3 \end{pmatrix} . \tag{37}$$

Using (35), (37), we get (see (27))

$$C_{3,1} = \tag{38}$$

$$= \frac{1}{1536}\begin{pmatrix} 18 & -150 & 900 & 900 & -150 & 18 \\ -33 & 197 & -1194 & 1194 & -197 & 33 \\ -20 & 156 & -136 & -136 & 156 & -20 \\ 57 & -317 & 666 & -666 & 317 & -57 \\ 2 & -6 & 4 & 4 & -6 & 2 \\ -31 & 155 & -310 & 310 & -155 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & -35 & 70 & -70 & 35 & -7 \end{pmatrix} .$$

With the symbol $\tilde{C}_{3,1}$, we mark the matrix which forms from the matrix $C_{3,1}$ (see (37)) by omitting the seventh row, composed of zeros.

Proceed to the calculation of parametric polynomials first and second segment of the opened interpolation curve $P_1P_2P_3P_4P_5$. According to (28) and (17), parametric equations of the first segment $P_1P_2 : -1 \le t \le 1$ are:

$$x_1 = P_1^{(1)}(t) = \tag{39}$$

$$= (1, t, t^2, t^3, t^4, t^5, t^7) \cdot \tilde{C}_{3,1} \cdot (5,2,1,2,5,2)^T =$$

$$= \frac{1}{192}(222 + 63t + 68t^2 + 57t^3 - 2t^4 - 31t^5 + 7t^7) ,$$

$$x_2 = P_2^{(1)}(t) = \tag{40}$$

$$= (1, t, t^2, t^3, t^4, t^5, t^7) \cdot \tilde{C}_{3,1} \cdot (-1,3,1,3,-1,-3)^T =$$

$$= \frac{1}{768}(1614 + 1555t - 76t^2 - 1243t^3 - 2t^4 + 589t^5 - 133t^7) .$$

$(P_1^{(1)}(-1), P_2^{(1)}(-1)) = (1,1) = P_1, (P_1^{(1)}(1), P_2^{(1)}(1)) = (2,3) = P_2$ is in accordance with the primary requirement.

According to (28) and (17), parametric equations of the second segment P_2P_3: $-1 \le t \le 1$ are:

$$x_1 = P_1^{(2)}(t) = \tag{41}$$

$$= (1, t, t^2, t^3, t^4, t^5, t^7) \cdot \tilde{C}_{3,1} \cdot (2,1,2,5,2,4)^T =$$

$$= \frac{1}{1536}(5958 + 3451t - 604t^2 - 1795t^3 + 22t^4 + 837t^5 - 189t^7) ,$$

$$x_2 = P_2^{(2)}(t) = (1, t, t^2, t^3, t^4, t^5, t^7) \cdot C_{3,1} \cdot (3,1,3,-1,-3,5)^T =$$

$$= \frac{1}{768}(1122 - 1961t - 372t^2 + 641t^3 + 18t^4 - 279t^5 + 63t^7) .$$

$(P_1^{(2)}(-1), P_2^{(2)}(-1)) = (2,3) = P_2, (P_1^{(2)}(1), P_2^{(2)}(1)) = (5,-1) = P_3$ is in accordance with the primary requirement.

Further applies

$$\frac{\mathrm{d}}{\mathrm{d}t} P_1^{(1)}(1) = \frac{\mathrm{d}}{\mathrm{d}t} P_1^{(2)}(-1) = \frac{4}{3} ,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} P_1^{(1)}(1) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} P_1^{(2)}(-1) = \frac{2}{3} ,$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3} P_1^{(1)}(1) = \frac{\mathrm{d}^3}{\mathrm{d}t^3} P_1^{(2)}(-1) = -\frac{1}{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} P_2^{(1)}(1) = \frac{\mathrm{d}}{\mathrm{d}t} P_2^{(2)}(-1) = -\frac{5}{12} ,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} P_2^{(1)}(1) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} P_2^{(2)}(-1) = -\frac{45}{24} ,$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3} P_2^{(1)}(1) = \frac{\mathrm{d}^3}{\mathrm{d}t^3} P_2^{(2)}(-1) = -\frac{1}{8} ,$$

thus, the transition from the first segment to the second segment of the constructed interpolation curve runs smoothly from the 1st to the 3rd derivation. This fact is also in accordance with the primary requirement.

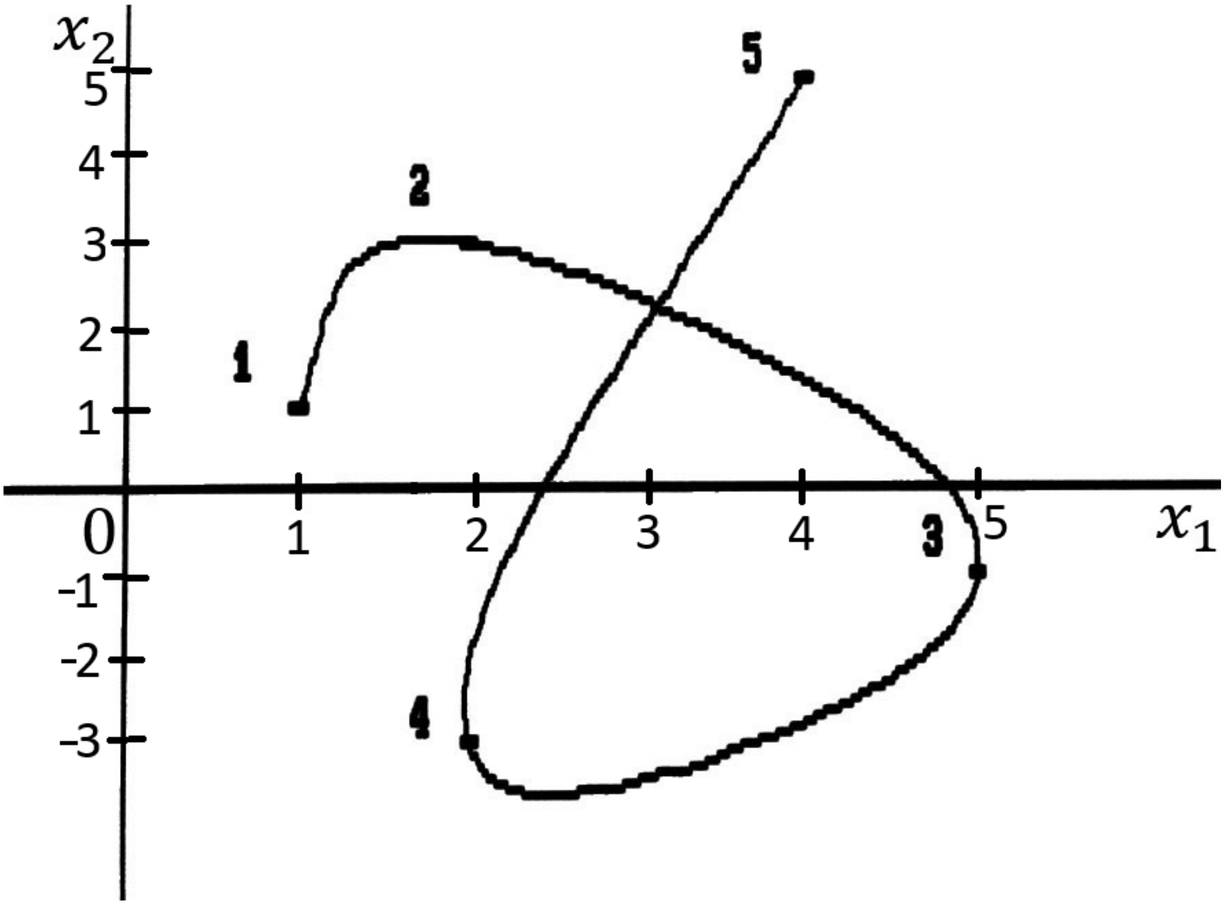

Fig. 2. The course of the opened interpolation curve $P_1P_2P_3P_4P_5$.

6 General Parameterisation of the Interpolation Curve

We recall that each segment of an opened interpolation curve $P_1P_2 \dots P_n$, or a closed interpolation curve $P_1P_2 \dots P_nP_1$, is parameterised on the same interval $< -1,1 >$. On the numerical axis, we now choose points $T_1 < T_2 < \dots < T_n$ in case of the opened curve $P_1P_2 \dots P_n$, or points $T_1 < T_2 < \dots < T_n < T_{n+1}$ in case of the closed curve $P_1P_2 \dots P_nP_1$, and the i-th interval $t \in< T_i, T_{i+1} >$ for $i = 1, \dots, n-1$ in case of the opened interpolation curve, or for $i = 1, \dots, n$ in case of the closed interpolation curve, we simply display it on the interval $< -1,1 >$ according to the formula

$$\frac{t-A_i}{B_i}, \text{ where } A_i = \frac{T_i+T_{i+1}}{2}, B_i = \frac{T_{i+1}-T_i}{2}. \tag{42}$$

For $\lambda \in< 0,1 >$, the point $t = T_i + \lambda(T_{i+1} - T_i)$ from the interval $< T_i, T_{i+1} >$ displays on the number $2\lambda - 1$. For instance, for $\lambda = 0$, the point T_i displays on the number -1, for $\lambda = 1$, the point T_{i+1} displays on the number 1, for $\lambda = ½$, the centre of the interval $< T_i, T_{i+1} >$ displays at zero.

For $j = 1, \ldots, m$, let

$$P_j^{(i)}(t) = (1, t, t^2, \ldots, t^{2Q+1}) \cdot C_{Q,p} \cdot \begin{pmatrix} x_j^{(i-Q+p)} \\ x_j^{(i-Q+p+1)} \\ \vdots \\ \vdots \\ x_j^{(i+Q-p+1)} \end{pmatrix},$$

where $t \in < -1,1 >$, are parametric polynomials of the i-th segment of the constructed interpolation curve (see (28)). Then for these parametric functions on the interval $t \in < T_i, T_{i+1} >$ applies the expression

$$P_j^{(i)}(t) = \tag{43}$$

$$= \left(1, \frac{t-A_i}{B_i}, (\frac{t-A_i}{B_i})^2, \ldots, (\frac{t-A_i}{B_i})^{2Q+1}\right) \cdot C_{Q,p} \cdot \begin{pmatrix} x_j^{(i-Q+p)} \\ x_j^{(i-Q+p+1)} \\ \vdots \\ \vdots \\ x_j^{(i+Q-p+1)} \end{pmatrix}.$$

7 Conclusion

The originality of Lienhard interpolation method is solely based on the given support points and with its assistance it generates all other quantities appearing in the mathematical formulation of interpolation problem. In contrast with the original version of Lienhard method, in which every segment of the interpolation curve output is parameterised on the same interval <-1,1>, the author of this article came up with the process of individual belonging of the parametric interval for each part of the interpolation curve.

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