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GENERALIZED LINEAR MODEL WITH SOFTWARE MATHEMATICA®

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Abstract. Generalized linear model is used to model the functional dependencies, where one variable is a random variable and the second variable is non-random.

Keywords: generalized linear model, vector of measurements, vector of parameters, design matrix, covariance matrix.

Mathematics Subject Classification: C1, C12, C6.

1 Introduction

In practice, there are many technical problems that are mathematically modeled as **Model 1**

$$Y_i = Y(x_i) = a_1 f_1(x_i) + a_2 f_2(x_i) + ... + a_k f_k(x_i) + e_i, \quad i = 1, 2, ..., n \tag{1}$$

where Y_i, $i = 1, 2, ..., n$ are random variables, whose realizations $y_1, y_2, ..., y_n$ are the measurement results.

These results depend on the function $a_1 f_1(x) + a_2 f_2(x) + ... + a_k f_k(x)$, known and non-random effects $x_1, x_2, ..., x_n$ and k parameters $a_1, a_2, \ldots, a_k$, whose values are unknown; e_i is a random error of the *i*-th measurement. Model 1 is called a **linear model** (linear in parameters).

Here are two examples from practice:

1. It is known from theory that in a certain range of temperatures x, the extension of the y copper pipe is a linear function of the temperature passing thru the null point. Measurements in this area can be written using a linear model

$$Y_i = a_1 x_i + e_i, \quad i = 1, 2, ..., n$$

2. Fuel consumption in a certain area of a car's speed is a quadratic function of speed. Therefore, the speed measurement x in this area can be described by a linear model

$$Y_i = a_1 + a_2 x_i + a_3 x_i^2 + e_i\,, \quad i = 1,\ 2,\ldots,\ n$$

2 Generalized linear model

Model 1 can be generalized and expressed in matrix form. We supposed, that

$$\boldsymbol{e} = (e_1,\ e_2,\ \ldots,\ e_n)^{\mathrm{T}} \tag{2}$$

is a vector of **random errors**, which is assumed to have the n-dimensional normal distribution with zero means:

$$E(e_i) = 0\,, \quad i = 1,\ 2,\ldots,n \tag{3}$$

and

$$\boldsymbol{\Sigma}_{\boldsymbol{e}} = \boldsymbol{\Sigma}_{\boldsymbol{Y}} = \sigma^2 \boldsymbol{H} \tag{4}$$

is a symmetric matrix of the type $n \times n$ called **covariance matrix**. $\boldsymbol{H}$ is a **matrix of weights** with elements h_{ij}, $i, j = 1, 2, \ldots, n$. The **variance** of the random variable Y_i is $\sigma^2 h_{ii} = \mathrm{cov}(Y_i, Y_i) = \sigma_i^2$ and the **covariance** between variables Y_i and Y_j is $\sigma^2 h_{ij} = \mathrm{cov}(Y_i, Y_j)$, $i \neq j$. Now we can write a generalized linear model in matrix form as **Model 2** as follows:

$$\boldsymbol{Y} = \boldsymbol{A\beta} + \boldsymbol{e}\,, \quad \boldsymbol{\Sigma}_{\boldsymbol{Y}} = \sigma^2 \boldsymbol{H} \tag{5}$$

where $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^{\mathrm{T}}$ is a known measurement vector,

$\boldsymbol{A}$ is known experimental design matrix of type $n \times k$,

$\boldsymbol{\beta} = (a_1,\ a_2,\ldots,\ a_k)^{\mathrm{T}}$ is a vector of unknown regression coefficients,

$\boldsymbol{e} = (e_1,\ e_2,\ \ldots,\ e_n)^{\mathrm{T}}$ is vector of random errors and

$\boldsymbol{\Sigma}_{\boldsymbol{Y}}$ is a covariance matrix.

Model 2 can be written as an ordered triple as follows:

$$(\boldsymbol{Y}, \boldsymbol{A\beta}, \sigma^2 \boldsymbol{H}) \tag{6}$$

Point estimators in Model

An estimator $\hat{\boldsymbol{\beta}}$ of the regression coefficients vector $\boldsymbol{\beta}$ is obtain by the generalized least squares method as the minimum of quadratic form

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{e} = (\boldsymbol{Y}-\boldsymbol{A\beta})^{\mathrm{T}}\boldsymbol{H}^{-1}(\boldsymbol{Y}-\boldsymbol{A\beta}) = f(\boldsymbol{\beta})$$

The quadratic form is adjusted to $f(\boldsymbol{\beta}) = \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{Y} - 2\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A\beta} + \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A\beta}$, and by deriving is to get $\frac{\partial f(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{Y} + 2\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A\beta}$.

It can be proved that the **least squares estimator of** $\hat{\boldsymbol{\beta}}$ is

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{Y} \tag{7}$$

This is the **minimum variance unbiased estimator (MVUE)** of each component a_i of the regression coefficients vector $\boldsymbol{\beta}$. Furthermore, it can be proved that the **covariance matrix of** $\hat{\boldsymbol{\beta}}$ is

$$\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \sigma^2\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{A}\right)^{-1} \tag{8}$$

and the **unbiased estimator of variance** σ^2 is

$$\hat{\sigma}^2 = \frac{(\boldsymbol{Y}-\boldsymbol{A}\hat{\boldsymbol{\beta}})^{\mathrm{T}}\boldsymbol{H}^{-1}(\boldsymbol{Y}-\boldsymbol{A}\hat{\boldsymbol{\beta}})}{n-k} \tag{9}$$

When (9) is inserted into (8) we get **the estimator of the covariance matrix of** $\hat{\boldsymbol{\beta}}$

$$\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = \hat{\sigma}^2\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1} \tag{10}$$

It is often necessary to estimate the various linear combinations of the parameter vector, which we call **parametric functions** and write in matrix form

$$\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta} = p_1 a_1 + p_2\, a_2 + \ ... + p_k\, a_k \tag{11}$$

where $\boldsymbol{p}^{\mathrm{T}} = (p_1,\ p_2,\ ...,\ p_k)$ and $\boldsymbol{\beta}^{\mathrm{T}} = (a_1, a_2, \ldots, a_k)$.

Let us consider the linear model $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{H})$ and the parametric function $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta}$. It can be proved that **the minimum variance unbiased estimator of the parametric function** is

$$\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} = \boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{Y} \tag{12}$$

and **the variance of the parametric function** is

$$D(\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}}) = \sigma^2 \boldsymbol{p}^{\mathrm{T}} \quad \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X} \quad^{-1} \boldsymbol{p} \tag{13}$$

When (9) is inserted into (13) we get the estimator of $D(\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}})$

$$D(\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2 \boldsymbol{p}^{\mathrm{T}} \quad \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X} \quad^{-1} \boldsymbol{p} \tag{14}$$

Assume that the vector $\boldsymbol{Y}$ in the linear models $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{H})$ has n-dimensional normal distribution then it is valid:

$$\frac{\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta}}{\sqrt{\sigma^2 \boldsymbol{p}^{\mathrm{T}} \quad \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X} \quad^{-1} \boldsymbol{p}}} \sim N(0,1) \tag{15}$$

$$\sigma^{-2}(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})^{T}\boldsymbol{H}^{-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) \sim \chi^2(n-k) \tag{16}$$

It can be proved that these two random variables are independent.
Then from the definition of *t*-distribution and relationship (9) it follows that

$$\frac{\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta}}{\sqrt{\hat{\sigma}^2 \boldsymbol{p}^{\mathrm{T}} \quad \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X} \quad^{-1} \boldsymbol{P}}} \sim t(n-k) \tag{17}$$

From the last statement, confidence intervals can be derived.

Confidence Intervals in Model 2

It is true that the $100(1-\alpha)$ % two-sided confidence interval for the parametric function $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta}$ is

$$\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} \pm t(n-k,\alpha)\sqrt{\frac{(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})^{\mathrm{T}}\boldsymbol{H}^{-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-k}\boldsymbol{p}^{\mathrm{T}} \quad \boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X} \quad^{-1} \boldsymbol{p}} \tag{18}$$

Using (18), it can be specify the interval estimate of any parameter (e.g. $a_1 = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta}$, where $\boldsymbol{p}^{\mathrm{T}} = (1,\ 0,\ ...,\ 0)$) and the functional value at any point.

When calculating estimates of the functional values at several points, we get the **confidence band around (theoretical) function** of $a_1 f_1(x) + a_2 f_2(x) + ... + a_k f_k(x)$. However, it cannot be interpreted this band in such a way that it in confidence $1-\alpha$ covers the entire theoretical function.
The confidence band for the entire function $a_1 f_1(x) + a_2 f_2(x) + ... + a_k f_k(x)$ covering this function with confidence $1-\alpha$ is given by the relationship

$$\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} \pm \sqrt{kF(k,n-k,\alpha)}\sqrt{\frac{(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})^{\mathrm{T}}\boldsymbol{H}^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-k}\boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{p}} \tag{19}$$

where $F(k,n-k,\alpha)$ is critical value of *F*–distribution with degrees of freedom k a $n-k$.

Tests of Hypotheses in Model 2

It is supposed to test the hypothesis H_0 that the actual value of the parametric function is equal to the real number q versus the alternative H_1 that it is not equal to this number. This hypothesis can be written as follows

$$H_0: \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta} = q \text{ vs. } H_1: \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\beta} \neq q \tag{20}$$

The test statistic, which has the *t*-distribution with $n-k$ degrees of freedom under the hypothesis H_0 validity, is given by the relation

$$T = \frac{\boldsymbol{p}^{\mathrm{T}}\hat{\boldsymbol{\beta}} - q}{\sqrt{\frac{(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})^{\mathrm{T}}\boldsymbol{H}^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})}{n-k}\boldsymbol{p}^{\mathrm{T}}\left(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{p}}} \tag{21}$$

The null hypothesis in (20) is rejected at the level of significance α when

$$|T| > t(n-k,\ \alpha) \tag{22}$$

These tests can be used to test arbitrary coefficient of the parameter vector, a functional value at any point, and so on. It is often used to test the statistical significance of individual regression coefficients. In this case, the hypothesis

$$H_0\colon\ a_i = 0 \tag{23}$$

is tested.

3 Example

It is given 4 independent measurements y_i with double precision at extreme points that are shown in Tab. 1.

x_i	2	3	4	5
y_i	3.5	1.7	1.3	2.6

Table 1. Independent measurements.

Assuming that the functional dependence between *x, y* is quadratic, we estimate the regression coefficients and their variance.

Solution

Since the functional dependence between *x, y* is quadratic, we are looking for the regression function $\hat{y} = \hat{a} + \hat{b}x + \hat{c}x^2 = \hat{a}x^0 + \hat{b}x + \hat{c}x^2$. It is necessary to calculate (estimate) unknown regression coefficients $\hat{a}, \hat{b}, \hat{c}$.

- For individual measured values y_i are :

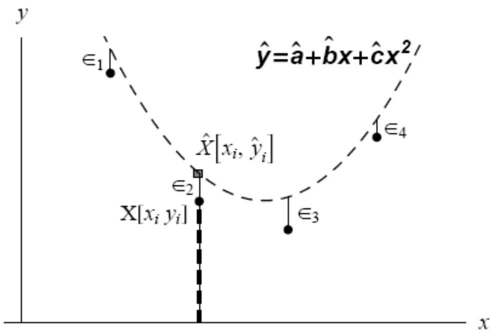

Fig. 1. Graphic individual measured values y_i.

in the form of equations:

$$y_i = \hat{y}_i + \varepsilon_i,\ i = 1,\dots,4$$

$$y_i = \hat{a}\cdot x_i^0 + \hat{b}\cdot x_i^1 + \hat{c}\cdot x_i^2 + \varepsilon_i, \quad i = 1,\dots,4$$

$$3,5 = \hat{a}\cdot 2^0 + \hat{b}\cdot 2^1 + \hat{c}\cdot 2^2 + \varepsilon_1$$

$$1,7 = \hat{a}\cdot 3^0 + \hat{b}\cdot 3^1 + \hat{c}\cdot 3^2 + \varepsilon_2$$

$$1,3 = \hat{a}\cdot 4^0 + \hat{b}\cdot 4^1 + \hat{c}\cdot 4^2 + \varepsilon_3$$

$$2,6 = \hat{a}\cdot 5^0 + \hat{b}\cdot 5^1 + \hat{c}\cdot 5^2 + \varepsilon_2$$

in matrix form:

$$\boldsymbol{y} \ = \ \boldsymbol{X} \ \times \ \boldsymbol{\beta} \ + \ \boldsymbol{\varepsilon}$$

$$\begin{pmatrix} 3,5 \\ 1,7 \\ 1,3 \\ 2,6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix} \times \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

where: $\boldsymbol{y}$ – measurement vector, $(\boldsymbol{Y}, \boldsymbol{X\beta}, \sigma^2\boldsymbol{H})$,
$\boldsymbol{X}$ – design matrix (of experiment),
$\boldsymbol{\varepsilon}$ – vector of random errors (vector of deviations between measured and estimated

- Covariance matrix of vectors $\boldsymbol{y}$ or $\boldsymbol{\varepsilon}$ is : $\boldsymbol{\Sigma_Y} = \boldsymbol{\Sigma_\varepsilon} = \sigma^2\boldsymbol{H} = \sigma^2 \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

where: σ^2 is unknown dispersion factor,
$\boldsymbol{H}$ is matrix of weights ($h_{11} = h_{44} = 2$ – double inaccuracy at extreme points).

- Regression coefficients $\hat{a}, \hat{b}, \hat{c}$ are estimated by the generalized least squares method (GLSM) – the sum of squares of measurement errors is required to be minimal.

For random errors ε_i of measurement y_i is valid:

in the form of equations:

$\varepsilon_i = y_i - \hat{y}_i, \; i = 1, \dots, 4$

$\varepsilon_i = y_i - \; \hat{a} \cdot x_i^0 + \hat{b} \cdot x_i^1 + \hat{c} \cdot x_i^2 \;\; , \quad i = 1, ..., 4$

$\varepsilon_1 = 3{,}5 - \; \hat{a} \cdot 2^0 + \hat{b} \cdot 2^1 + \hat{c} \cdot 2^2$

$\varepsilon_2 = 1{,}7 - \; \hat{a} \cdot 3^0 + \hat{b} \cdot 3^1 + \hat{c} \cdot 3^2$

$\varepsilon_3 = 1{,}3 - \; \hat{a} \cdot 4^0 + \hat{b} \cdot 4^1 + \hat{c} \cdot 4^2$

$\varepsilon_2 = 2{,}6 - \; \hat{a} \cdot 5^0 + \hat{b} \cdot 5^1 + \hat{c} \cdot 5^2$

in matrix form:

$\boldsymbol{\varepsilon} \;=\; \boldsymbol{y} \;-\; \boldsymbol{X} \;\times\; \boldsymbol{\beta}$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix} = \begin{pmatrix} 3{,}5 \\ 1{,}7 \\ 1{,}3 \\ 2{,}6 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix} \cdot \begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{pmatrix}$$

It is required, that $\boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{H}^{-1} \boldsymbol{\varepsilon} = (\boldsymbol{Y} - \boldsymbol{X\beta})^{\mathrm{T}} \boldsymbol{H}^{-1} (\boldsymbol{Y} - \boldsymbol{X\beta}) = \min$.

Here it should be noted that

- known: $\boldsymbol{y}$, $\boldsymbol{X}$ – design matrix, $\boldsymbol{H}$ − matrix of weights,
- unknown: $\boldsymbol{\beta} = [a, b, c]^T$ – wants to be estimated.

Therefore

$$\varphi(\boldsymbol{\beta}) = (\boldsymbol{Y} - \boldsymbol{X\beta})^{\mathrm{T}} \boldsymbol{H}^{-1} (\boldsymbol{Y} - \boldsymbol{X\beta})$$

$$\varphi([\hat{a}, \hat{b}, \hat{c}]^T) = (\boldsymbol{Y} - \boldsymbol{X\beta})^{\mathrm{T}} \boldsymbol{H}^{-1} (\boldsymbol{Y} - \boldsymbol{X\beta})$$

Minimize $\boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{H}^{-1} \boldsymbol{\varepsilon}$ actual means to find a local minimum vector $[\hat{a}, \hat{b}, \hat{c}]^T$ of the function $\varphi(\boldsymbol{\beta})$.

The function $\varphi(\boldsymbol{\beta})$ acquires a minimum at a stationary point $[\hat{a}, \hat{b}, \hat{c}]^T$ that is a solution to a system of normal equations

$$\frac{\partial \; \varphi(\boldsymbol{\beta})}{\partial \; \boldsymbol{\beta}} = \boldsymbol{0} \;\Leftrightarrow\; \frac{\partial \varphi(\boldsymbol{\beta})}{\partial \hat{a}} = 0 \wedge \frac{\partial \varphi(\boldsymbol{\beta})}{\partial \hat{b}} = 0 \wedge \frac{\partial \varphi(\boldsymbol{\beta})}{\partial \hat{c}} = 0 \, .$$

By solving the system of normal equations an estimate of the individual regression coefficients is obtained:

$$\boldsymbol{\beta} = \; \hat{a}, \hat{b}, \hat{c} \;^{T} \; = \; \boldsymbol{X}^T \boldsymbol{H}^{-1} \boldsymbol{X} \;^{-1} \cdot \; \boldsymbol{X}^T \boldsymbol{H}^{-1} \boldsymbol{y}$$

$$\hat{\beta}=\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}=\begin{pmatrix}\frac{1165}{22} & \frac{-1365}{44} & \frac{17}{4}\\ \frac{-1365}{44} & \frac{1649}{88} & \frac{-21}{8}\\ \frac{17}{4} & \frac{-21}{8} & \frac{3}{8}\end{pmatrix}\cdot\begin{pmatrix}6,05\\ 20,3\\ 75,6\end{pmatrix}=\begin{pmatrix}11,9136\\ -5,74318\\ 0,775\end{pmatrix}$$

Estimated regression function: $\hat{y}=\hat{a}+\hat{b}x+\hat{c}x^2=11,9136-5,74318x+0,775x^2$

Note.

The solution $\boldsymbol{\beta}$ exists only if the inverse matrix $\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}\ ^{-1}$ exists, that means the matrix $\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}$ is regular $\Rightarrow$ its determinant is different from zero ($\left|\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}\right|\neq 0$).

Auxiliary calculations (symmetric matrixes):

$$\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{y}=\begin{pmatrix}1 & 1 & 1 & 1\\ 2 & 3 & 4 & 5\\ 4 & 9 & 16 & 25\end{pmatrix}\cdot\begin{pmatrix}0,5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0,5\end{pmatrix}\cdot\begin{pmatrix}3,5\\ 1,7\\ 1,3\\ 2,6\end{pmatrix}=\begin{pmatrix}6,05\\ 20,3\\ 75,6\end{pmatrix}$$

symmetric matrixes:

$$\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}=\begin{pmatrix}1 & 1 & 1 & 1\\ 2 & 3 & 4 & 5\\ 4 & 9 & 16 & 25\end{pmatrix}\cdot\begin{pmatrix}0,5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0,5\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & 4\\ 1 & 3 & 9\\ 1 & 4 & 16\\ 1 & 5 & 25\end{pmatrix}=\begin{pmatrix}3 & 10,5 & 39,5\\ 10,5 & 39,5 & 157,5\\ 39,5 & 157,5 & 657,5\end{pmatrix}$$

$$\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}\ ^{-1}=\begin{pmatrix}\frac{1165}{22} & \frac{-1365}{44} & \frac{17}{4}\\ \frac{-1365}{44} & \frac{1649}{88} & \frac{-21}{8}\\ \frac{17}{4} & \frac{-21}{8} & \frac{3}{8}\end{pmatrix}=\begin{pmatrix}52,9545 & -31,0227 & 4,25\\ -31,0227 & 18,7386 & -2,625\\ 4,25 & -2,625 & 0,375\end{pmatrix}$$

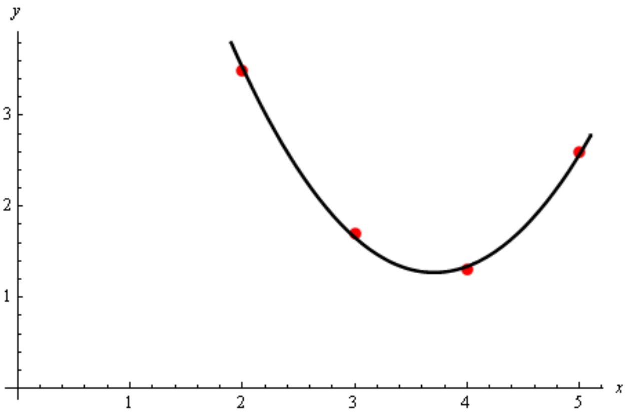

Fig. 2. Graphic verification of accuracy of the regression function found.

- To estimate the dispersion of regression coefficients, it is first necessary to calculate the dispersion estimate of σ^2:

$$\hat{\sigma}^2 = \frac{(\boldsymbol{y} - \boldsymbol{X\beta})^T \boldsymbol{H}^{-1} (\boldsymbol{y} - \boldsymbol{X\beta})}{n-k} = \frac{1}{n-k} (\boldsymbol{y} - \boldsymbol{X\beta})^T \boldsymbol{H}^{-1} (\boldsymbol{y} - \boldsymbol{X\beta})$$

where: n = 4 – number of measurements,
k = 3 – number of regression coefficient.

Then

$$\hat{\sigma}^2 = \frac{1}{4-3} (\boldsymbol{y} - \boldsymbol{X\beta})^T \boldsymbol{H}^{-1} (\boldsymbol{y} - \boldsymbol{X\beta}) = (\boldsymbol{y} - \boldsymbol{X\beta})^T \boldsymbol{H}^{-1} (\boldsymbol{y} - \boldsymbol{X\beta})$$

$$\hat{\sigma}^2 = \begin{pmatrix} -0,0272727 \\ 0,0409091 \\ -0,0409091 \\ 0,0272727 \end{pmatrix}^T \cdot \begin{pmatrix} 0,5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0,5 \end{pmatrix} \cdot \begin{pmatrix} -0,0272727 \\ 0,0409091 \\ -0,0409091 \\ 0,0272727 \end{pmatrix}$$

$$\hat{\sigma}^2 = 0,00409091$$

The estimator of the covariance matrix of $\hat{\boldsymbol{\beta}}$ (symmetric matrix) is:

$$\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = \hat{\sigma}^2 \quad \boldsymbol{X}^{\mathrm{T}} \boldsymbol{H}^{-1} \boldsymbol{X} \quad ^{-1}$$

$$\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = 0,00409091 \cdot \begin{pmatrix} \frac{1165}{22} & \frac{-1365}{44} & \frac{17}{4} \\ \frac{-1365}{44} & \frac{1649}{88} & \frac{-21}{8} \\ \frac{17}{4} & \frac{-21}{8} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 0,216632 & -0,126911 & 0,017386 \\ -0,126911 & 0,076658 & -0,010738 \\ 0,017386 & -0,010738 & 0,001534 \end{pmatrix}$$

Variances of the estimated regression coefficients lie on the main diagonal of the matrix and their values are:

$$D(\hat{a}) = 0,216632$$

$$D(\hat{b}) = 0,0766581$$

$$D(\hat{c}) = 0,00153409$$

4 Conclusion

The article presents the theory of the generalized linear model. The example shows the use of a diagonal covariance matrix in the model. The end points are measured with double precision.

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