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PARAMETER ESTIMATION FOR SCALAR STOCHASTIC DIFFERENTIAL EQUATION OF SECOND ORDER

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Abstract. We introduce (scalar) stochastic differential equation of second order with two unknown parameters and its solution, which is Ornstein–Uhlenbeck process. Based on ergodicity, we derive two families of the strong consistent estimators of these parameters. Then we study their asymptotic normality and we present their implementation.

Keywords: Ornstein–Uhlenbeck process, parameter estimation, asymptotic normality

Mathematics subject classification: Primary 62M05; Secondary 60G35, 93E10.

1 Introduction

In this article, we are improving and extending some previous results on the parameter estimation for stochastic differential equation of second order, which were previously published in [3] and [2]. Namely, we have found another family of estimators $(\tilde{a}_t, \tilde{b}_t)$, we have proved their asymptotic normality and we have implemented them in the program R.

The following section introduces the stochastic differential equation for the harmonic oscillation, its rewriting and its solution, that is the Ornstein–Uhlenbeck process. Using the observation of the trajectory $\{X^{x_0}(t),\, 0 \leq t \leq T\}$ and ergodicity of the Ornstein–Uhlenbeck process, two types of strong constistent estimators of unknown parameters are proposed. The proof of Theorem 1 may be found in [4]. The proof of Lemma 1 is only a matter of computation. We also mention the asymptotic normality of the estimators $\hat{a}_T$ and $\hat{b}_T$, which was studied in [2].

The asymptotic normality of the estimators $\tilde{a}_T$ and $\tilde{b}_T$ is proved in the third section. We are using Itô's formula to obtain different formulae for the processes $Y_T = \int_0^T |X_1^{x_0}(t)|^2\, dt$ and $H_T = \int_0^T |X_2^{x_0}(t)|^2\, dt$ (on which the estimators $\tilde{a}_T$ and $\tilde{b}_T$ are based on) and then central limit theorem for the stochastic integral (which may be found in [5]). The results are summarized in Theorem 3.

In the section 4, we introduce the implementation of all estimators and on one particular example, we compare the two methods both graphically and numerically.

2 Parameter estimation and strong consistency

Consider the following (scalar) stochastic differential equation

$$\ddot{x} + 2a\dot{x} + bx = \sigma\dot{\beta}(t), \tag{1}$$

with initial values $X(0) = x_1$ and $\dot{X}(0) = x_2$. Let $a > 0$, $b > 0$ be unknown real parameters, $\sigma > 0$ is known and let $\dot{\beta}(t)$ be the formal time derivative of the standard Brownian motion.

We may rewrite this equation in the form

$$dX(t) = AX(t)\,dt + \Phi\,dB(t), \quad X(0) = x_0, \tag{2}$$

if we set

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \in \mathbb{R}^2,\ A = \begin{pmatrix} 0 & 1 \\ -b & -2a \end{pmatrix} \in \mathbb{M}_{2\times 2},\ \Phi = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix} \in \mathbb{M}_{2\times 2},$$

$$B(t) = \begin{pmatrix} 0 \\ \beta(t) \end{pmatrix} \in \mathbb{R}^2,\ x_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

where $\mathbb{M}_{2\times 2}$ stands for the space of all 2×2 matrices with real entries, equipped with the operator norm $\|\cdot\|_{\mathcal{L}(\mathbb{R}^2)}$.

If we denote $S(t) := e^{At}$ a strongly continuous semigroup on $\mathbb{R}^2$, then the solution $(X^{x_0}(t), t \geq 0)$ to (2) is defined by the mild form

$$X^{x_0}(t) = S(t)x_0 + Z(t), \quad t \geq 0, \tag{3}$$

where $(Z(t), t \geq 0)$ is the convolution integral

$$Z(t) = \int_0^t S(t-u)\Phi\,dB(u). \tag{4}$$

The solution $(X^{x_0}(t), t \geq 0)$ to the equation (2) is called the Ornstein–Uhlenbeck process.

Theorem 1. *If the semigroup $(S(t), t \geq 0)$ is exponentially stable, i.e., there exist constants $M > 0$ and $\rho > 0$ such that for all $t \geq 0$*

$$\|S(t)\|_{\mathcal{L}(\mathbb{R}^2)} \leq Me^{-\rho t}, \tag{A1}$$

then there is a Gaussian centered limiting measure $\mu_\infty^{(a,b)} = N(0, Q_\infty^{(a,b)})$ for $(X^{x_0}(t), t \geq 0)$, such that

$$w^* - \lim_{t\to\infty} \mu_t^{x_0} = \mu_\infty^{(a,b)}$$

for each initial condition $x_0 \in \mathbb{R}^2$, where $\mu_t^{x_0} = \mathrm{Law}(X^{x_0}(t))$ and $\mathrm{Law}(\cdot)$ denotes the probability distribution. The covariance matrix $Q_\infty^{(a,b)}$ has the following form

$$Q_\infty^{(a,b)} = \int_0^\infty S(t)\Phi\Phi^\top S^\top(t)\,dt. \tag{5}$$

If $a^2 < b$, then the real parts of eigenvalues of matrix A are negative and the condition (A1) holds true. If we denote $\alpha = -a$, $\beta = \sqrt{b - a^2}$, then we may write $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$.

Lemma 1. The fundamental system $(S(t), t \geq 0)$ for the equation (1) has the following form

$$S(t) = \left(\begin{array}{cc} e^{\alpha t}\left(\cos(\beta t) - \frac{\alpha}{\beta}\sin(\beta t)\right) & \frac{1}{\beta}e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\left(-\beta - \frac{\alpha^2}{\beta}\right)\sin(\beta t) & \frac{1}{\beta}e^{\alpha t}(\alpha\sin(\beta t) + \beta\cos(\beta t)) \end{array} \right).$$

The covariance matrix $Q_\infty^{(a,b)}$ of the limiting measure $\mu_\infty^{(a,b)}$ equals to

$$\begin{aligned} Q_\infty^{(a,b)} &= \int_0^\infty S(t)\Phi\Phi^\top S^\top(t)\,dt = \int_0^\infty \frac{\sigma^2}{\beta^2}\left(\begin{array}{cc} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{array} \right) dt = \frac{\sigma^2}{\beta^2}\left(\begin{array}{cc} \frac{\beta^2}{4ab} & 0 \\ 0 & \frac{\beta^2}{4a} \end{array} \right) \\ &= \sigma^2 \left(\begin{array}{cc} \frac{1}{4ab} & 0 \\ 0 & \frac{1}{4a} \end{array} \right), \end{aligned} \tag{6}$$

where

$$\begin{aligned} q_{11}(t) &= e^{2\alpha t}\sin^2(\beta t), \\ q_{12}(t) &= q_{21}(t) = e^{2\alpha t}\sin(\beta t)(\alpha\sin(\beta t) + \beta\cos(\beta t)), \\ q_{22}(t) &= e^{2\alpha t}(\alpha\sin(\beta t) + \beta\cos(\beta t))^2. \end{aligned}$$

Remark 1. In the cases of $a^2 > b$ and $a^2 = b$, the eigenvalues of matrix A are also negative and the condition (A1) also holds true. The formulae for the fundamental system $(S(t), t \geq 0)$ are different, but the resulting formulae for the covariance matrix $Q_\infty^{(a,b)}$ coincide. Hence the assumption $a^2 < b$ may be omitted.

For the parameter estimation, we are interested in the trace of the covariance matrix $Q_\infty^{(a,b)}$. From the formula (6), it follows that

$$\operatorname{Tr} Q_\infty^{(a,b)} = \frac{b+1}{4ab}\sigma^2. \tag{7}$$

Since the Ornstein–Uhlenbeck process $X(t)$ is ergodic in $\mathbb{R}^2$ (see Example 2.1. in [7]), we may use Birkhoff's theorem. Namely

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_{\mathbb{R}^2}^2\,dt = \int_{\mathbb{R}^2} \|y\|_{\mathbb{R}^2}^2\,d\mu_\infty^{(a,b)}(y) = \operatorname{Tr} Q_\infty^{(a,b)}, \tag{8}$$

for any initial contidion $x_0 \in \mathbb{R}^2$ (see Theorem 4.9. in [6]).

If we denote $I_T := \frac{1}{T}\int_0^T \|X^{x_0}(t)\|_{\mathbb{R}^2}^2\,dt$, then (based on (7)) some strongly constistent estimators of parameters a and b may be proposed. If we know the true value of the parameter b, then the strongly consistent estimator of the parameter a is

$$\hat{a}_T = \frac{b+1}{4bI_T}\sigma^2. \tag{9}$$

Similarly, if we know the true value of the parameter a, then

$$\hat{b}_T = \frac{\sigma^2}{4aI_T - \sigma^2} \tag{10}$$

is the strong consistent estimator of the parameter b.

In [2], we have also proved the asymptotic normality of these estimators.

Theorem 2. *The estimators $\hat{a}_T$ and $\hat{b}_T$ are asymptotically normal, i.e.,*

$$\operatorname{Law}\left(\sqrt{T}\left(\hat{a}_T - a\right)\right) \to N\left(0,\ \frac{4a^3}{b(b+1)^2} + a\right), \quad T \to \infty, \tag{11}$$

$$\operatorname{Law}\left(\sqrt{T}\left(\hat{b}_T - b\right)\right) \to N\left(0,\ 4ab + \frac{b^2(b+1)^2}{a}\right), \quad T \to \infty. \tag{12}$$

The estimators $\hat{a}_T$ and $\hat{b}_T$ are indeed very easily implemented, but they have one major disadvantage: We have to know the true value of the other parameter. However another family of estimators $(\tilde{a}_T, \tilde{b}_T)$ may be proposed, which do not possess this disadvantage. Since

$$\|x\|^2_{\mathbb{R}^2} = |x_1|^2 + |x_2|^2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

the integral in (8) may be split into two parts

$$\begin{aligned} I_T &= \frac{1}{T}\int_0^T \|X^{x_0}(t)\|^2_{\mathbb{R}^2}\, dt \\ &= \frac{1}{T}\int_0^T |X_1^{x_0}(t)|^2\, dt + \frac{1}{T}\int_0^T |X_2^{x_0}(t)|^2\, dt \\ &=: Y_T + H_T, \end{aligned} \tag{13}$$

where $X^{x_0}(t) = (X_1^{x_0}(t), X_2^{x_0}(t))^\top \in \mathbb{R}^2$ is the solution to the equation (2). The formula (7) for the $\operatorname{Tr} Q_\infty^{(a,b)}$ may also be split into two parts. Indeed, the two parts in (13) converge to their corresponding limits individually, since it is just using the Birkhoff's theorem to the individual components, i.e.,

$$\lim_{T\to\infty} Y_T = \lim_{T\to\infty} \frac{1}{T}\int_0^T |X_1^{x_0}(t)|^2\, dt = \frac{\sigma^2}{4ab}, \tag{14}$$

$$\lim_{T\to\infty} H_T = \lim_{T\to\infty} \frac{1}{T}\int_0^T |X_2^{x_0}(t)|^2\, dt = \frac{\sigma^2}{4a}. \tag{15}$$

Based on these convergences, we may introduce new family of strongly consistent estimators $\tilde{a}_T$ and $\tilde{b}_T$

$$\tilde{a}_T = \frac{\sigma^2}{4H_T}, \tag{16}$$

$$\tilde{b}_T = \frac{H_T}{Y_T}. \tag{17}$$

3 Asymptotic normality of estimators $\tilde{a}_T$ and $\tilde{b}_T$

In this section, we show asymptotic normality of estimators (16) and (17), i.e., the weak convergence of $\sqrt{T}\left(\tilde{a}_T - a\right)$ or $\sqrt{T}\left(\tilde{b}_T - b\right)$ to a Gaussian distribution.

Let us start with the definition of operators $P_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $P_2 : \mathbb{R}^2 \to \mathbb{R}^2$:

$$P_1 x = P_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \tag{18}$$

$$P_2 x = P_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b + 4a^2 & 2a \\ 2a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \tag{19}$$

Let us denote $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, the Euclidean scalar product and the norm in $\mathbb{R}^2$. The properties of these two matrices are summarized in the following Lemma 2.

Lemma 2. The matrices P_1 and P_2 are symmetric and

$$\langle P_1 x, Ax\rangle = -2a|x_2|^2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \tag{20}$$

$$\langle P_2 x, Ax\rangle = -2ab|x_1|^2, \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2. \tag{21}$$

Proof. The symmetry of matrices P_1 and P_2 is evident. For every $x = (x_1, x_2)^\top \in \mathbb{R}^2$, we have

$$\begin{aligned}
\langle P_1 x, Ax\rangle &= \left\langle \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -b & -2a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle \\
&= \left\langle \begin{pmatrix} bx_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ -bx_1 - 2ax_2 \end{pmatrix}\right\rangle \\
&= bx_1x_2 - bx_1x_2 - 2ax_2^2 \\
&= -2a|x_2|^2
\end{aligned}$$

and

$$\begin{aligned}
\langle P_2 x, Ax\rangle &= \left\langle \begin{pmatrix} b+4a^2 & 2a \\ 2a & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -b & -2a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\rangle \\
&= \left\langle \begin{pmatrix} bx_1 + 4a^2x_1 + 2ax_2 \\ 2ax_1 + x_2 \end{pmatrix}, \begin{pmatrix} x_2 \\ -bx_1 - 2ax_2 \end{pmatrix}\right\rangle \\
&= bx_1x_2 + 4a^2x_1x_2 + 2ax_2^2 - 2abx_1^2 - 4a^2x_1x_2 - bx_1x_2 - 2ax_2^2 \\
&= -2ab|x_1|^2.
\end{aligned}$$

□

We will also need the alternative representations for the processes Y_T and H_T, which were defined by (13).

Lemma 3. The process Y_T admits the following representation

$$\begin{aligned}
Y_T &= \frac{1}{T}\int_0^T |X_1^{x_0}(t)|^2\,dt \\
&= -\frac{1}{4abT}\left(\langle P_2 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_2 x_0, x_0\rangle\right) + \frac{1}{2abT}\int_0^T \langle P_2 X^{x_0}(t), \Phi\,dB(t)\rangle + \frac{\sigma^2}{4ab}.
\end{aligned} \tag{22}$$

The process H_T admits the following representation

$$\begin{aligned}
H_T &= \frac{1}{T}\int_0^T |X_2^{x_0}(t)|^2\,dt \\
&= -\frac{1}{4aT}\left(\langle P_1 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_1 x_0, x_0\rangle\right) + \frac{1}{2aT}\int_0^T \langle P_1 X^{x_0}(t), \Phi\,dB(t)\rangle + \frac{\sigma^2}{4a}.
\end{aligned} \tag{23}$$

Proof. Define the function $g_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$g_1(x) = \langle P_1 x, x\rangle\,, \quad \forall x \in \mathbb{R}^2. \tag{24}$$

The application of Itô's formula to the function $g_1(X^{x_0}(t))$ yields

$$dg_1(X^{x_0}(t)) = 2\left\langle P_1 X^{x_0}(t), dX^{x_0}(t)\right\rangle + \frac{1}{2}\operatorname{Tr}\left(2P_1\Phi\Phi^\top\right)\,dt. \tag{25}$$

The second term may be simplified via following calculation

$$\frac{1}{2}\operatorname{Tr}\left(2P_1\Phi\Phi^\top\right) = \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2.$$

The expression (25) may be now written in the following way

$$\begin{aligned} dg_1(X^{x_0}(t)) &= 2\left\langle P_1 X^{x_0}(t), \mathcal{A}X^{x_0}(t)\right\rangle\,dt + 2\left\langle P_1 X^{x_0}(t), \Phi\,dB(t)\right\rangle + \sigma^2\,dt \\ &= -4a|X_2^{x_0}(t)|^2\,dt + 2\left\langle P_1 X^{x_0}(t), \Phi\,dB(t)\right\rangle + \sigma^2\,dt. \end{aligned}$$

After integrating previous formula over the interval $(0,T)$ and after some algebraic operations, we will arrive at (23).

Similarly, if we define function $g_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$g_2(x) = \langle P_2 x, x\rangle\,, \quad \forall x \in \mathbb{R}^2 \tag{26}$$

and apply Itô's formula to the function $g_2(X^{x_0}(t))$, we will obtain

$$dg_2(X^{x_0}(t)) = 2\left\langle P_2 X^{x_0}(t), dX^{x_0}(t)\right\rangle + \frac{1}{2}\operatorname{Tr}\left(2P_2\Phi\Phi^\top\right)\,dt. \tag{27}$$

Since the second term equals to

$$\frac{1}{2}\operatorname{Tr}\left(2P_2\Phi\Phi^\top\right) = \operatorname{Tr}\begin{pmatrix} 0 & 2a\sigma^2 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2,$$

the formula (27) and Lemma 2 yield

$$\begin{aligned} dg_2(X^{x_0}(t)) &= 2\left\langle P_2 X^{x_0}(t), \mathcal{A}X^{x_0}(t)\right\rangle\,dt + 2\left\langle P_2 X^{x_0}(t), \Phi\,dB(t)\right\rangle + \sigma^2\,dt \\ &= -4ab|X_1^{x_0}(t)|^2\,dt + 2\left\langle P_2 X^{x_0}(t), \Phi\,dB(t)\right\rangle + \sigma^2\,dt. \end{aligned}$$

After integrating previous formula over the interval $(0,T)$ and after some algebraic operations, we will arrive at (22). □

3.1 Asymptotic normality of the estimator $\tilde{a}_T$

Using formula (16) for the estimator $\tilde{a}_T$ and formula (23) for H_T, we are able to compute the following

$$\begin{aligned} &\sqrt{T}\,(\tilde{a}_T - a) = \sqrt{T}\left(\frac{\sigma^2}{4H_T} - a\right) = \sqrt{T}\,\frac{\sigma^2 - 4aH_T}{4H_T} \\ &= \frac{\sqrt{T}}{4H_T}\left(\frac{1}{T}\left(\langle P_1 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_1 x_0, x_0\rangle\right) - \frac{2}{T}\int_0^T \langle P_1 X^{x_0}(t), \Phi\,dB(t)\rangle\right) \\ &= \frac{1}{4H_T}\frac{1}{\sqrt{T}}\left(\langle P_1 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_1 x_0, x_0\rangle\right) - \frac{1}{2H_T}\frac{1}{\sqrt{T}}\int_0^T \langle P_1 X^{x_0}(t), \Phi\,dB(t)\rangle\,. \end{aligned} \tag{28}$$

Using Chebyshev's inequality, it is easy to show, that the first term $\frac{1}{\sqrt{T}}\left(\langle P_1 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_1 x_0, x_0\rangle\right) \to 0$ in probability as $T \to \infty$. We may also write

$$Z_1(T) := \frac{1}{\sqrt{T}} \int_0^T \langle P_1 X^{x_0}(t), \Phi\, dB(t)\rangle = \frac{1}{\sqrt{T}} \int_0^T \sum_{n=1}^{2} \langle P_1 X^{x_0}(t), e_n\rangle \; d\,\langle e_n, \Phi B(t)\rangle$$
$$= \frac{\sigma}{\sqrt{T}} \int_0^T \langle P_1 X^{x_0}(t), e_2\rangle \; d\beta(t) = \frac{\sigma}{\sqrt{T}} \int_0^T X_2^{x_0}(t)\, d\beta(t), \tag{29}$$

where we have used that $\Phi B(t) = (0, \sigma\beta(t))^\top$.

By the central limit theorem for the stochastic integral (see Proposition 1.22. in [5]), $Z_1(T)$ converges weakly to a Gaussian distribution with a zero mean and variance given by

$$\lim_{T\to\infty} \frac{\sigma^2}{T} \int_0^T \left(X_2^{x_0}(t)\right)^2 \, dt = \sigma^2\, \mathbb{E}\left(X_2(\infty)\right)^2 = \sigma^2\, \mathrm{Var}\left(X_2(\infty)\right) = \frac{\sigma^4}{4a}, \tag{30}$$

where $X(\infty) = (X_1(\infty), X_2(\infty))^\top$ is an $\mathbb{R}^2$–valued Gaussian random variable with zero mean and covariance matrix $Q_\infty^{(a,b)}$. (It has the invariant distribution $\mu_\infty^{(a,b)}$.)

Since the multiplicative factor $-\frac{1}{2H_T}$ of $Z_1(T)$ in (28) converges to $-\frac{2a}{\sigma^2}$ as $T \to \infty$, we have the following results

$$\mathrm{Law}\left(Z_1(T)\right) \to N\left(0, \frac{\sigma^4}{4a}\right), \; T \to \infty, \tag{31}$$
$$\mathrm{Law}\left(\sqrt{T}\left(\tilde{a}_T - a\right)\right) \to N\left(0,\, a\right), \; T \to \infty. \tag{32}$$

3.2 Asymptotic normality of the estimator $\tilde{b}_T$

Using formula (17) for the estimator $\tilde{b}_T$ and Lemma 3 for representation of Y_T and H_T, we are able to compute the following

$$\sqrt{T}\left(\tilde{b}_T - b\right) = \sqrt{T}\left(\frac{H_T}{Y_T} - b\right) = \frac{\sqrt{T}}{Y_T}\left(H_T - bY_T\right)$$
$$= \frac{\sqrt{T}}{Y_T}\left(-\frac{1}{4aT}\left(\langle P_1 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_1 x_0, x_0\rangle\right) + \frac{1}{2aT}\int_0^T \langle P_1 X^{x_0}(t), \Phi\, dB(t)\rangle\right.$$
$$\left. + \frac{1}{4aT}\left(\langle P_2 X^{x_0}(T), X^{x_0}(T)\rangle - \langle P_2 x_0, x_0\rangle\right) - \frac{1}{2aT}\int_0^T \langle P_2 X^{x_0}(t), \Phi\, dB(t)\rangle\right)$$
$$= \frac{1}{4aY_T}\frac{1}{\sqrt{T}}\left(\langle (P_2 - P_1) X^{x_0}(T), X^{x_0}(T)\rangle - \langle (P_2 - P_1) x_0, x_0\rangle\right)$$
$$- \frac{1}{2aY_T}\frac{1}{\sqrt{T}}\int_0^T \langle (P_2 - P_1) X^{x_0}(t), \Phi\, dB(t)\rangle\,. \tag{33}$$

Similarly as above, the term

$$\frac{1}{\sqrt{T}}\left(\langle (P_2 - P_1) X^{x_0}(T), X^{x_0}(T)\rangle - \langle (P_2 - P_1) x_0, x_0\rangle\right) \to 0$$

in probability as $T \to \infty$. We may also compute

$$
\begin{aligned}
Z_2(T) &:= \frac{1}{\sqrt{T}} \int_0^T \langle (P_2 - P_1) X^{x_0}(t), \Phi \, dB(t) \rangle \\
&= \frac{1}{\sqrt{T}} \int_0^T \sum_{n=1}^{2} \langle (P_2 - P_1) X^{x_0}(t), e_n \rangle \; d \langle e_n, \Phi B(t) \rangle \\
&= \frac{\sigma}{\sqrt{T}} \int_0^T \langle (P_2 - P_1) X^{x_0}(t), e_2 \rangle \; d\beta(t) = \frac{2a\sigma}{\sqrt{T}} \int_0^T X_1^{x_0}(t) \, d\beta(t), \qquad (34)
\end{aligned}
$$

since $\langle (P_2 - P_1) X^{x_0}(t), e_2 \rangle = 2a X_1^{x_0}(t)$. Similarly to $Z_1(T)$, $Z_2(T)$ also converges weakly to a Gaussian distribution with a zero mean and variance given by

$$
\lim_{T \to \infty} \frac{4a^2\sigma^2}{T} \int_0^T \left(X_1^{x_0}(t) \right)^2 \, dt = 4a^2\sigma^2 \, \mathbb{E} \left(X_1(\infty) \right)^2 = 4a^2\sigma^2 \operatorname{Var} \left(X_1(\infty) \right) = \frac{a\sigma^4}{b}. \qquad (35)
$$

The multiplicative factor $-\frac{1}{2aY_T}$ of $Z_2(T)$ in (33) converges to $-\frac{2b}{\sigma^2}$ as $T \to \infty$, which brings us to the following results

$$
\operatorname{Law} \left(Z_2(T) \right) \to N \left(0, \, \frac{a\sigma^4}{b} \right), \, T \to \infty, \qquad (36)
$$

$$
\operatorname{Law} \left(\sqrt{T} \left(\tilde{b}_T - b \right) \right) \to N \left(0, \, 4ab \right), \, T \to \infty. \qquad (37)
$$

We may summarize the results in the following theorem.

Theorem 3. *The estimators $\tilde{a}_T$ and $\tilde{b}_T$ are asymptotically normal, i.e.,*

$$
\begin{aligned}
\operatorname{Law} \left(\sqrt{T} \left(\tilde{a}_T - a \right) \right) &\to N \left(0, \, a \right), \, T \to \infty, \\
\operatorname{Law} \left(\sqrt{T} \left(\tilde{b}_T - b \right) \right) &\to N \left(0, \, 4ab \right), \, T \to \infty.
\end{aligned}
$$

The family of estimators $(\tilde{a}_T, \tilde{b}_T)$ is also better than the family $(\hat{a}_T, \hat{b}_T)$ in the sense that its limiting variances are smaller. Indeed, if we compare (32) to (11), we see that the limiting variance of $\hat{a}_T$ is $a + \frac{4a^3}{b(b+1)^2}$ and the limiting variance of $\tilde{a}_T$ is just a. Similarly, if we compare (37) to (12), we recognize that the limiting variance of $\hat{b}_T$ is $4ab + \frac{b^2(b+1)^2}{a}$ and the limiting variance of $\tilde{b}_T$ is just $4ab$. The estimators $(\tilde{a}_T, \tilde{b}_T)$ are strict upgrade to the estimators $(\hat{a}_T, \hat{b}_T)$.

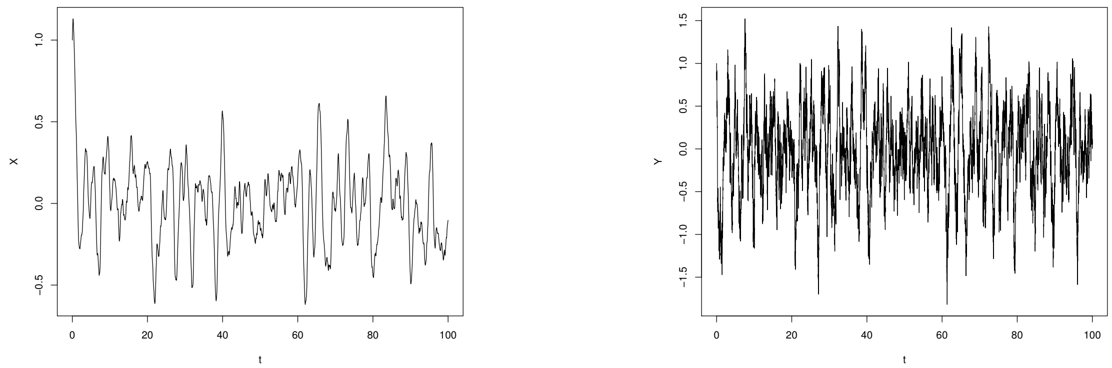
4 Implementation and statistical evidence

We have generated a trajectory of the solution to the stochastic differential equation (1) by Euler's method (see for example [1]). We have chosen $T = 100$ (the length of the time interval), $\Delta t = 0,01$ (the mesh of the partition), $x_1 = 1$, $x_2 = 1$ (the initial values), $a = 1$, $b = 4$ and $\sigma = 1$. The implementation in R code is as follows.

```
T <- 100
N <- 10000
Delta <- T/N
t <- seq(0, T, length = N+1)
```

```
X <- numeric(N+1)
Y <- numeric(N+1)
X0 <- 1
Y0 <- 1
a <- 1
b <- 4
sigma <- 1
X[1] <- X0
Y[1] <- Y0
set.seed(123)
Z <- rnorm(N)
for (i in 2:(N+1)){
X[i] <- X[i-1] + Y[i-1]*Delta
Y[i] <- Y[i-1] + (- b*X[i-1] - 2*a*Y[i-1])*Delta + sigma*Z[i-1]*sqrt(Delta)
}
plot(t, X, type = "l")
plot(t, Y, type = "l")
```

The following figure shows the solution to the equation (1) (that is the process $X_1^{x_0}(t)$) and its derivative (that is the process $X_2^{x_0}(t)$).



(a) The process $X_1^{x_0}(t)$.

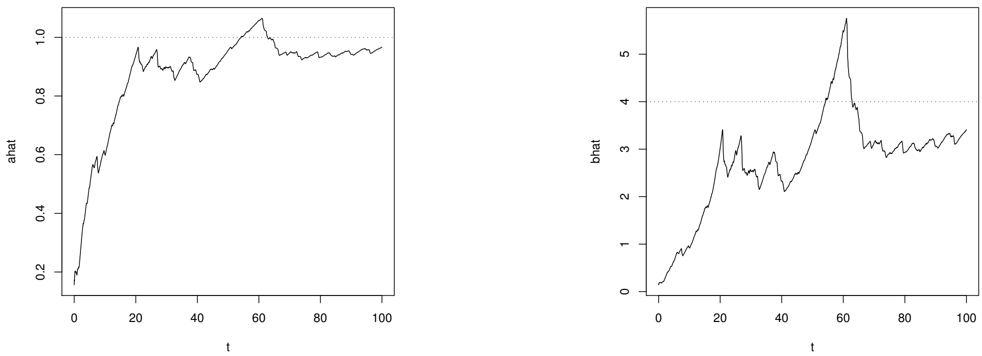
(b) The process $X_2^{x_0}(t)$.

Fig. 1

The implementation of estimators $\hat{a}_T$ and $\hat{b}_T$ is described by the following code.

```
I <- numeric(N+1)
ahat <- numeric(N+1)
bhat <- numeric(N+1)
I[1] <- X[1]^2 + Y[1]^2
ahat[1] <- sigma^2*(b+1)/(4*b*I[1])
bhat[1] <- sigma^2/(4*a*I[1] - sigma^2)
for (i in 2:(N+1)){
I[i] <- (I[i-1]*(i-1) + X[i]^2 + Y[i]^2)/i
ahat[i] <- sigma^2*(b+1)/(4*b*I[i])
bhat[i] <- sigma^2/(4*a*I[i] - sigma^2)
}
```

The value of the statistic I_T (on which the estimators $\hat{a}_T$ and $\hat{b}_T$ are based on (see (8))) equals to $I_T = 0,3233$, while the trace of the matrix $Q_\infty^{(a,b)}$ equals to $Q_\infty^{(a,b)} = \frac{b+1}{4ab}\sigma^2 = 0,3125$. The estimators of a and b are $\hat{a}_T = 0,9667$, $\hat{b}_T = 3,4122$ and their time evolution is shown in the following Figure.



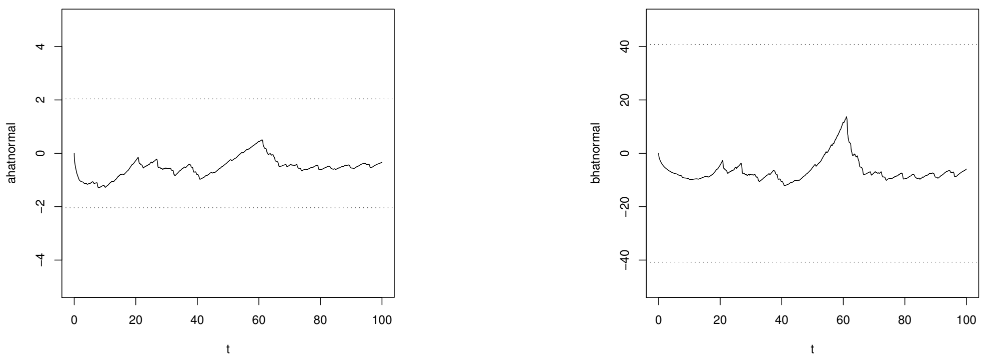
(a) The estimator $\hat{a}_t$. (b) The estimator $\hat{b}_t$.

Fig. 2

The implementation of formulae from Theorem 2 is as follows.

```
ahatnormal <- sqrt(t)*(ahat - a)
bhatnormal <- sqrt(t)*(bhat - b)
vara <- a + (4*a^3)/(b*(b+1)^2)
2*sqrt(vara)
varb <- 4*a*b + (b^2*(b+1)^2)/a
2*sqrt(varb)
```

The limiting variance from the formula (11) equals to $1,04$ and the limiting variance from the formula (12) equals to 416. The Figure 3 shows the progress of variables $\sqrt{t}(\hat{a}_t - a)$ and $\sqrt{t}(\hat{b}_t - b)$, where boundaries for "the rule of 2σ" are depicted. (Normally distributed random variable $X \sim N(\mu, \sigma^2)$ is realized in the interval $(\mu - 2\sigma, \mu + 2\sigma)$ with $95,45\%$ probability.)



(a) The process $\sqrt{t}(\hat{a}_t - a)$. (b) The process $\sqrt{t}(\hat{b}_t - b)$.

Fig. 3

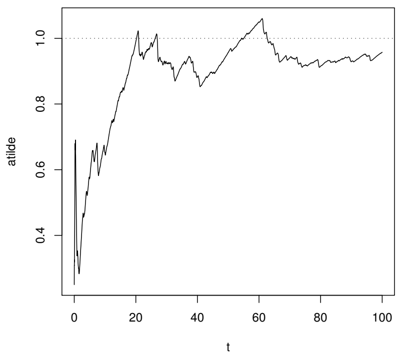
All the pictures seem very satisfactory, however let us introduce the estimators $\tilde{a}_T$ and $\tilde{b}_T$. Their implementation is also very simple.

```
I2X <- numeric(N+1)
I2Y <- numeric(N+1)
atilde <- numeric(N+1)
btilde <- numeric(N+1)
I2X[1] <- X[1]^2
I2Y[1] <- Y[1]^2
atilde[1] <- sigma^2/(4*I2Y[1])
btilde[1] <- I2Y[1]/I2X[1]
for (i in 2:(N+1)){
I2X[i] <- (I2X[i-1]*(i-1) + X[i]^2)/i
I2Y[i] <- (I2Y[i-1]*(i-1) + Y[i]^2)/i
atilde[i] <- sigma^2/(4*I2Y[i])
btilde[i] <- I2Y[i]/I2X[i]
}
```

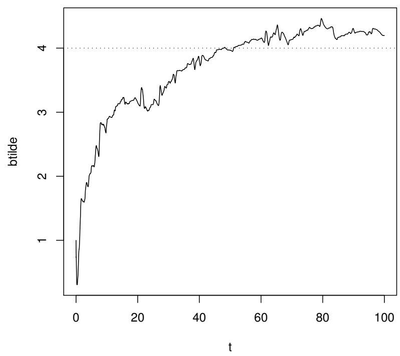
The results are as follows

$$\begin{aligned} Y_T &= 0,0622, & \frac{\sigma^2}{4ab} &= 0,0625, \\ H_T &= 0,2611, & \frac{\sigma^2}{4a} &= 0,25, \\ \tilde{a}_T &= 0,9576, & \tilde{b}_T &= 4,1964. \end{aligned}$$

Time evolution of the estimators $\tilde{a}_T$ and $\tilde{b}_T$ is shown in the following figure.



(a) The estimator $\tilde{a}_t$.

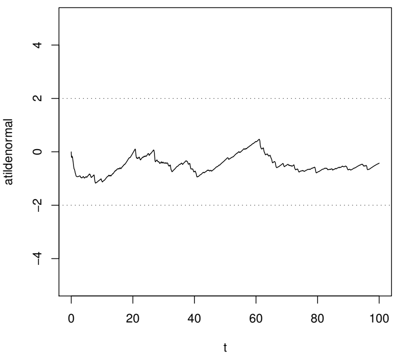


(b) The estimator $\tilde{b}_t$.

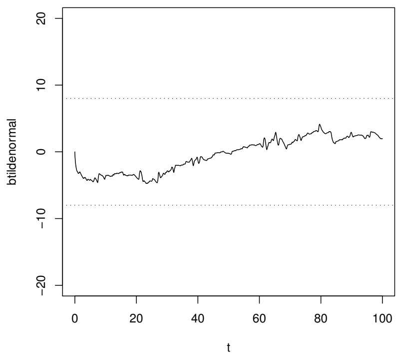
Fig. 4

The implementation of the formulae (32) and (37) is similar as above. Let us recall, that the limiting variance of the estimator $\tilde{a}_T$ is $a = 1$ and limiting variance of the estimator $\tilde{b}_T$ is $4ab = 16$. The following figure shows the progress of variables $\sqrt{t}(\tilde{a}_t - a)$ and $\sqrt{t}(\tilde{b}_t - b)$ as well as the 95, 45% confidence interval.

>From the previous (concrete) simulation it follows, that although the estimator $\tilde{a}_T$ is (locally) worse than the estimator $\hat{a}_T$, the estimator $\tilde{b}_T$ is much more better than $\hat{b}_T$. After running many and many simulations (also with different parameters a, b, σ, x_1, x_2, T, Δt), we claim that all estimators have their derived properties and that our implementation is correct and fully functional.



(a) The process $\sqrt{t}(\tilde{a}_t - a)$.



(b) The process $\sqrt{t}(\tilde{b}_t - b)$.

Fig. 5

5 Conclusion

We have introduced the stochastic differential equation of second order and its solution, which is Ornstein–Uhlenbeck process. Based on ergodicity, two families of the strong consistent estimators of unknown parameters have been derived.

In the second part of the paper, we have proved the asymptotic normality of the family $(\tilde{a}_T, \tilde{b}_T)$ and we have shown that this family of estimators is better than the family $(\hat{a}_T, \hat{b}_T)$ (e.g., it is possible to use them without any knowledge of the true value of the other parameter and their limiting variances are smaller).

The third part of the paper presents the implementation of used methods and their comparison on one concrete example.

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