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# SERLIO´S, GUARINI´S AND MEYER´S CONSTRUCTIONS OF OVALS IN ARCHITECTURE

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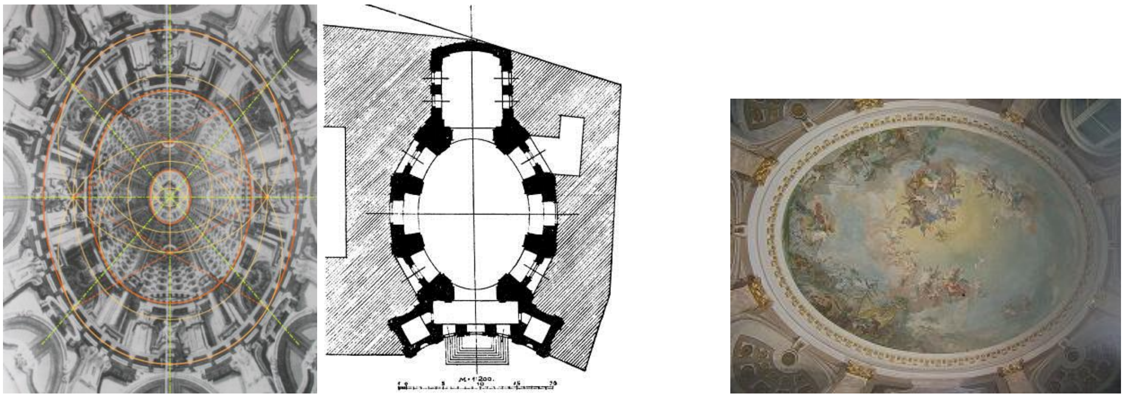
**Abstract.** When reconstructing some of the buildings with elliptical ground plan, it is often very difficult to unambiguously determine whether an ellipse or an oval was constructed to approximate this ellipse. We deal with Serlio´s and Guarini´s constructions of ovals. F. S. Meyer showed constructions of ovals whose gives very good approximation of the ellipse. We modify Serlio´s and Guarini´s constructions of ovals so that lengths of both axes of the ellipse are preserved.

**Keywords:** an oval, a construction, approximation of an ellipse

*Mathematics Subject Classification*: Primary 51E21, 51N05; Secondary 65D17

## 1 Introduction

Ellipse is a form that is found common in architecture. F. S. Meyer in his book A Hand Book of Ornament [6] said: “The Ellipse is of comparatively late appearance in art, the construction presupposing a certain knowledge of Geometry, which was not possessed by primitive peoples. The ellipse is a very popular shape for ceilings, panels, boxes and dishes.”
Exact construction of ellipse is difficult because its radius of curvature is continually changing. In practice, therefore, it is approximated by circular arcs and the expression "Oval" is often used for the ellipse, but it is erroneous.
The word oval derived from the Latin word “ovus” for egg. Here it is used in terms of the shape of curves that resemble an egg. It is a curve for which strict mathematical definition is missing but has the common features: is a closed curve in a plane which "loosely" resembles the outline of an egg, is differentiable (smooth-looking), convex, closed, simple (not self-intersecting) plane curve, which has at least one axis of symmetry. In a special case and also in architecture, we see the oval as a curve, which is composed of circular arcs. We can find such an approach in books of authors as S. Serlio [1], G. Guarini [2] and F. S. Meyer [6].
We are interested in the study and geometric analysis of buildings with elements of ovals. These are buildings that have ground oval or elliptic plan or ovals have been used in the construction of theirs internal building elements. In Slovakia we can for example show buildings as Cathedral of St John of Matha in Bratislava and elliptic St. Ladislav Chapel in the former Ostrihom archbishop´s palace in Bratislava (Fig.1)



(a) (b) (c)

Fig. 1. Cathedral of St John of Matha (a), (b), Ostrihom archbishop´s palace (c).

Architects approximate the ellipse by circular arcs (oval) in practice. Due to the symmetry of an ellipse it is enough to use two different circles most often. We search the centers and radii of the two circles that are connected smoothly. The circles have a common tangent at the point of connection, which means that the centers *H*, *K* (Fig. 2) of these circles must belong on one line. Let $h = |SH|$ and $k = |SK|$ where *S* is the center of the oval. Found circles have radii $a - h, a - h + \sqrt{h^2 + k^2}$ where *a* is the length of semi-major axis of the ellipse.

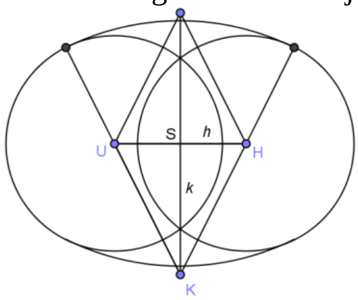
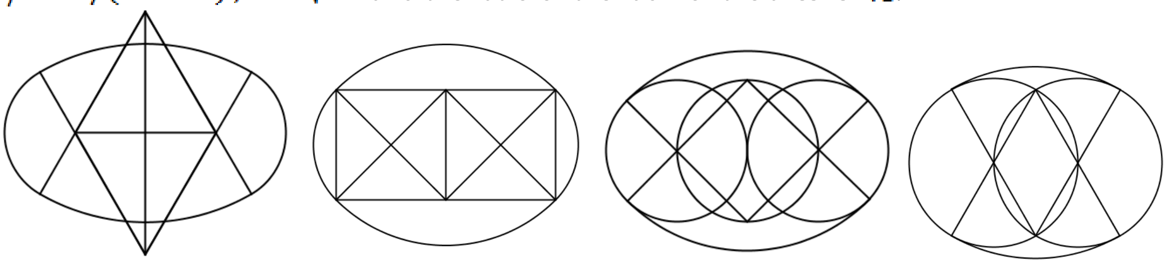



Fig. 2. Construction of two-center oval, based upon Serlio´s treatise.

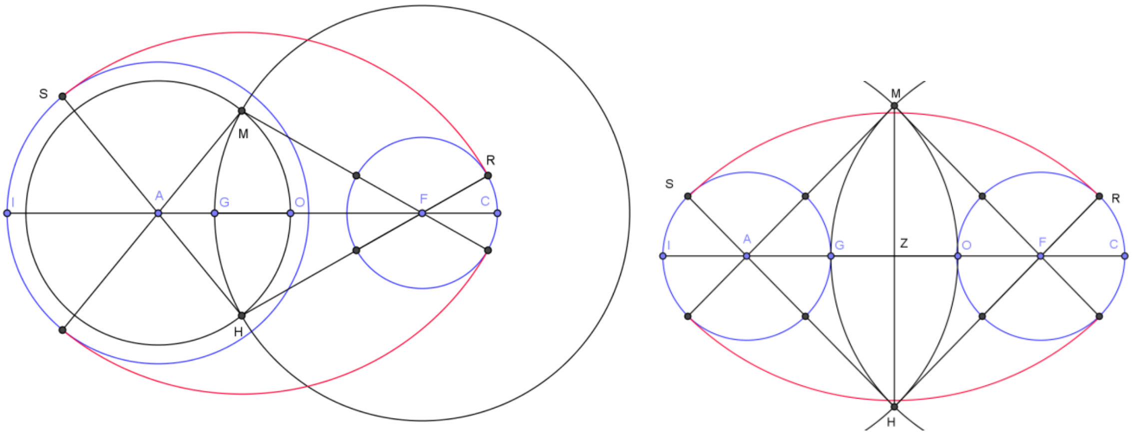
In paper [3], P. L. Rosin dealt with Serlio´s constructions (Fig. 3). The construction of the Fig. 3(a) is in the case that $\Delta UHK$ is the equilateral triangle and $h = (a - b)/(\sqrt{3} - 1)$, $k = \sqrt{3}h$. The ratio of the radius of the circles, and the ratio $a/b$ are not constant. These ratios are constant in other Serlio's construction. In case Fig. 3(b) is $h = k = a/(\sqrt{2} + 1)$, $a/b = (\sqrt{2} + 1)/(2\sqrt{2} - 1)$ and the ratio of the radii of the arcs is ½. In case Fig. 3(c) is $h = k = a/2$, $a/b = \sqrt{2}$ and the ratio of the radii of the arcs is $\sqrt{2} - 1$. In case Fig. 3(d) is $h = a/3$, $a/b = 3/(4 - \sqrt{3})$, $k = \sqrt{3}h$ and the ratio of the radii of the arcs is ½.



(a) (b) (c) (d)

Fig. 3. Oval constructions by S. Serlio.

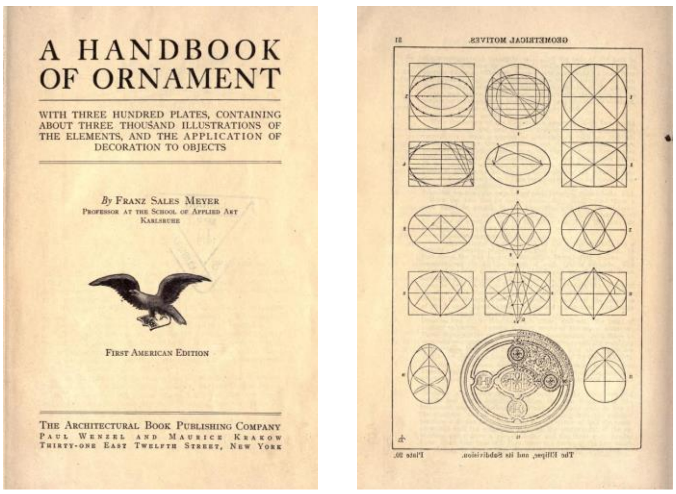
We have dealt with the ovals in architecture in the articles [4], [5]. These are constructions that approximate the ellipse while observe the length of the major axis of the ellipse. G. Guarini is describing in [2] construction to an oval generally.
We can take two circles with centers *A* and *F*, which have any distance, non-intersecting or intersecting (each other), with equal or unequal radii. We take non-intersecting circles (Fig. 4(a)) with different radii. Line *AF* intersects these circles in points *I* and *C*. We construct points *G* and *O*, such that $|IO|$ equals $|CG|$ and it is equal to the length greater than half the line segment *CI* length. We have circles with centers *A* and *F* and radii *AO*, *GF*. These circles have common the points *M* and *H*. It is clear that the line *MH* is perpendicular to the line *AF*. We get lines *MF*, *HF*, *MA*, *HA*. The lines intersect the circles at points *S*, *R*. (Fig. 4), points *S* and *R* lie on a circle with center *H*.




(a) (b)

Fig. 4. Ovals constructed using Guarini´s method.

By appropriate choice of the characteristics of Guarini's oval we can get Serlio's oval. For example, we get Serlio's oval in the construction in Fig. 3(d) so that we choose the radii of the circles as $2a/3$, the distance between centers of the circles is $2a/3$ and the point *G* is identical to the point *A*.




(a) (b)

Fig. 5. F. S. Meyer, A Handbook of Ornament (a), Plate 20, pp. 31 (b).

## 2 Meyer´s constructions

Franz Sales Meyer in his book A Handbook of Ornament [6] introduced the modified Serlio´s and Guarini´s constructions. Unlike these two authors, he showed constructions where not only the length of the major axis of the ellipse but also the length of minor axis of the ellipse is preserved.

First we will show the construction by which F.S. Meyer in [6, Plate 20, pp. 31, no. 8] modified the Serlio's construction (c) (see Fig. 5 (b)). One center of the approximation circle is point $S_2$. For Serlio's construction it is point K. In Fig. 6 it is a comparison of these two constructions.

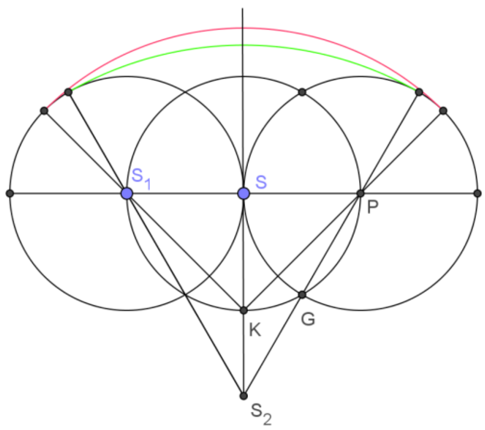



Fig. 6. Meyer´s construction (green colour), Serlio´s construction (c) (red colour).

It is clear that the triangle $\Delta\, S_2S_1P$ is equilateral and hence lengths are $h = a/2$, $k = \sqrt{3}a/2$, $a/b = (\sqrt{3}+3)/3$ and the ratio of radii of the circles is 1/3.

We have now the ellipse given by its axes. We are dealing with the construction, which is mentioned in [6, Plate 20, pp. 31, no. 11].
Point $S$ is the center of the ellipse. We take half the difference between the semi-major axis and the semi-minor axis of the ellipse. From the center $S$ we apply this distance three times to the major axis and four times to the minor axis. We obtain centers $S_1$, $S_2$ of circles that we will use to approximate the ellipse, see Fig. 7.

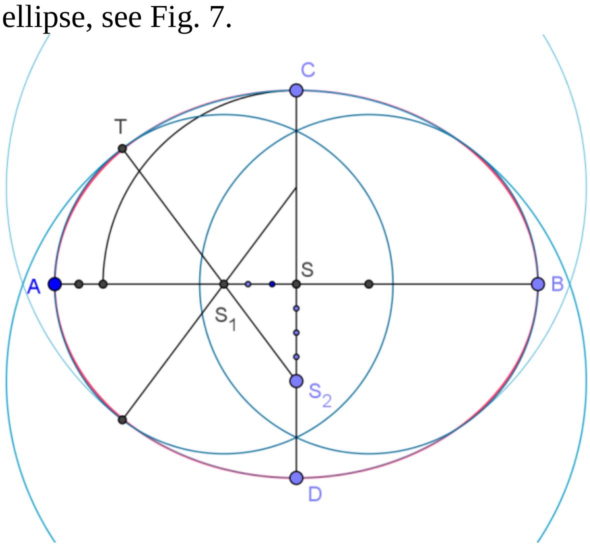



Fig. 7

We will denote a distance between a center of a circle on the major axis and the center of the ellipse as $h = |S_1S|$ and a distance of a center of a circle on the minor axis from the center of the ellipse as $k = |SS_2|$ as with Serlio´s and Guarini´s constructions.
In this case is $|SA| = a$, $|SC| = b$, $h = 3(a-b)/2$, $k = 2(a-b)$, $h = 3k/4$ and radii of circles are $r_1 = (3b-a)/2$, $r_2 = 2a-b$. We still have to show that $|S_2T| = |S_2C|$. Based on the construction is $|S_1S_2| = 5(a-b)/2$. $|S_2T| = |S_1S_2|+|S_1T| = 5(a-b)/2 + (3b-a)/2 = 2a-b = |S_2C|$. Ellipse (red colour) in Fig. 7 has $a/b = 5/4$ and its approximation according to the construction just described.

We construct the major and minor axes of the given ellipse as in [6, Plate 20, pp. 31, no. 10]. Point $S$ is the center of the ellipse. We take the difference between the semi-major axis and the semi-minor axis of the ellipse. We construct a line segment $AC$ and apply the length $a-b$ from the point $C$ to this line segment and we get a point $Q$. We mark the center of the line segment $AQ$ as a point $L$. We draw the perpendicular to the line $AC$ through the point $L$. This line intersects the major axis at a point $S_1$ and the minor axis at a point $S_2$, which are centers of circles that approximate these ellipse. Based on the construction, the triangles $\Delta\, ALS_1$, $\Delta\, S_2LC$, $\Delta\, ASC$, $\Delta\, S_2SS_1$ are similar to each other (Fig. 8) and the length

$$|AL| = \frac{\sqrt{a^2+b^2}-(a-b)}{2}.$$

From the similarity $\Delta\, ALS_1 \approx \Delta\, ASC$ follows that

$$r_1 = |AS_1| = \frac{|AL||AC|}{|AS|} = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}-(a-b)\right)}{2a}.$$

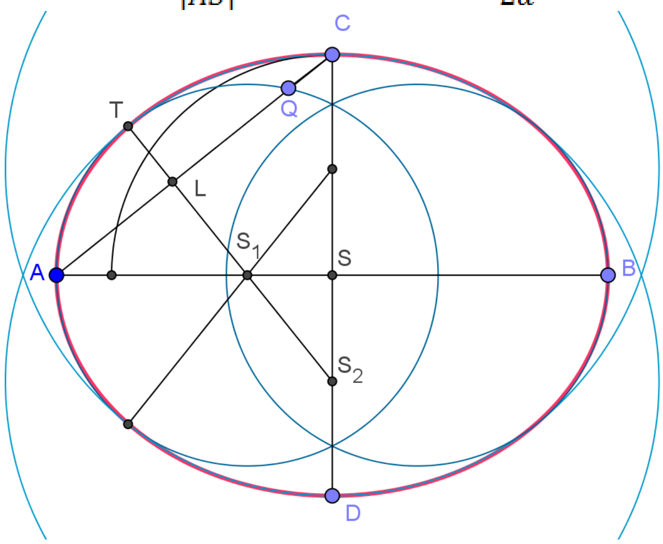


Fig. 8

We express length $h = |SS_1| = a - |AS_1| = (a-b)(\sqrt{a^2+b^2}+a+b)/(2a)$.
The similarity $\Delta\, ASC \approx \Delta\, S_2SS_1$ implies $|SS_1|/|SC| = |S_2S|/|AS|$ and $k = |S_2S| = a|SS_1|/b = ah/b$. From the same similarity, we have $|SS_1|/|SC| = |S_2S_1|/|AC|$, since $|S_2S_1| = \sqrt{a^2+b^2}\,h/b$. The similarity $\Delta\, S_2LC \approx \Delta\, ASC$ gives

$$r_2 = |CS_2| = \frac{|CL||AC|}{|CS|} = \frac{\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2}+(a-b)\right)}{2b}.$$

We must prove that it is $|S_2T| = |S_2C|$. Based on the construction it is $|S_2T| = |S_1S_2|+|S_1T| = \sqrt{a^2+b^2}\,h/b + |AL|\sqrt{a^2+b^2}/a = \sqrt{a^2+b^2}(\sqrt{a^2+b^2}+(a-b))/(2b) = |S_2C|$. In Fig. 8 is the ellipse with $a/b = 5/4$ (red colour) and its approximation by circular arcs.

We see that both constructions very well approximate this ellipse. We now show constructions that use the osculating circles of the ellipse and the ellipse is approximated by an oval determined by three different circles.

F. S. Meyer showed in [6, Plate 20, pp. 31, no. 12] a smooth connecting of osculating circles of the ellipse with another circular arc.

We construct a rectangle *ORPQ* (Fig. 9) whose sides have a length equal to the major and minor axis of the ellipse and a center of the rectangle is the center *S* of the ellipse and the axes of its sides are line segments *AB*, *CD*. We draw a line segment *AC*. In the point *Q* draw a perpendicular line to the line *AC*. This line crosses the major axis in point $S_1$ and the minor axis in point $S_2$, which are the centers of osculating circles.

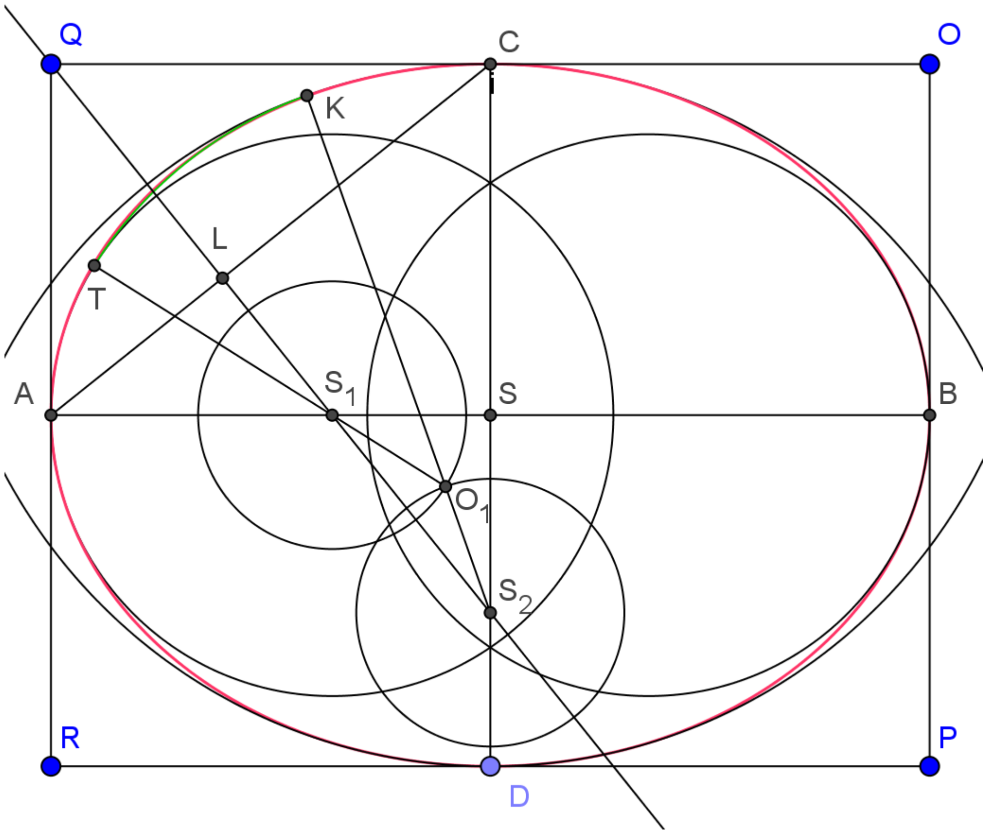



Fig. 9

The radius of the osculating circles is known, but we can easily deduce them from the similarity of triangles $\Delta\, ALS_1$, $\Delta\, S_2LC$, $\Delta\, ASC$, $\Delta\, S_2SS_1$, $r_1 = |AS_1| = b^2/a$, $r_2 = |CS_2| = a^2/b$. The similarity $\Delta\, ASC \approx \Delta\, S_2SS_1$ gives $|SS_1|/|SC| = |S_2S|/|AS|$ and it is $k = |S_2S| = a|SS_1|/b = ah/b$, where $h = a - r_1 = (a^2 - b^2)/a$.

We draw circles with centers at points $S_1$, $S_2$ and with the radius $(r_2 - r_1)/2$. In their intersection is the center $O_1$ of the circle (green colour), which smoothly feeds the osculating circles into the continuous curve - oval. The lines $O_1S_1$, $O_1S_2$ intersect osculating circles at points *T*, *K*. We must prove that $|TO_1| = |KO_1|$. By construction we have

$$|TO_1| = (r_2 - r_1)/2 + r_1 = (r_2 + r_1)/2 \text{ and } |KO_1| = r_2 - (r_2 - r_1)/2 = (r_2 + r_1)/2.$$

Although this construction is an approximate construction, the constructed oval is the best approximation of the ellipse from all of these constructions. In Fig. 9 is the ellipse with $a/b$ = 5/4 (red colour).

## 3 Modification of Serlio´s and Guarini´s constructions

If we give the condition for constructions of S. Serlio and G. Guarini that we want to approximate the ellipse with known the major and the minor axes, then the following equality must be satisfied

$$r_1 + \sqrt{h^2 + k^2} = k + b, \tag{1}$$

where $h = a - r_1$ and $r_1$ is a radius of a circle $q_1$, which approximates the ellipse in its vertex. If we calculate unknown $k$ from the equation (1) we will get

$$k = \frac{(a-b)}{2}\frac{(a+b-2r_1)}{b-r_1} \tag{2}$$

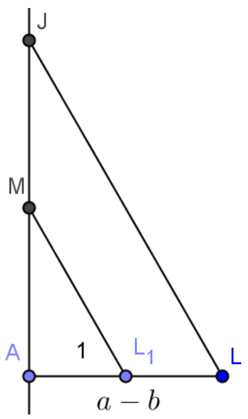
Based on the procedure in Serlio´s construction (a), where we try to construct the equilateral triangle $\Delta\, S_2S_1O_1$ (Fig. 10 (b)) is $k = \sqrt{3}h$ and therefore $r_1 = a - k/\sqrt{3}$.
After inserting into equation (2) we get the quadratic equation

$$k^2 - (a-b)\left(1+\sqrt{3}\right)k + \frac{\sqrt{3}}{2}(a-b)^2 = 0,$$

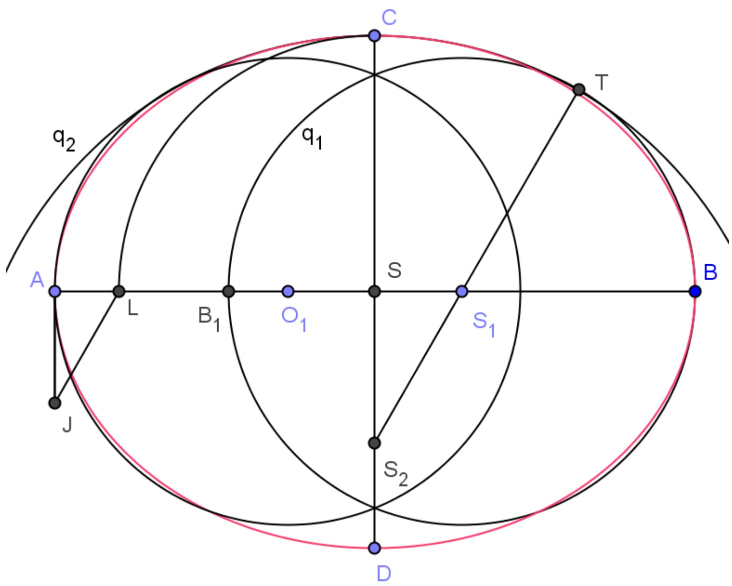
whose solutions are $k_1 = (a-b)\left(3+\sqrt{3}\right)/2$, $\bar{k}_1 = (a-b)\left(\sqrt{3}-1\right)/2$. From this we get the expression for the radius $r_1$ approximating circle $q_1$ for the vertex of an ellipse

$$r_1 = \frac{a\left(1-\sqrt{3}\right)+b\left(1+\sqrt{3}\right)}{2}, \quad \bar{r}_1 = \frac{a+b}{2} + \frac{(a-b)}{2\sqrt{3}}.$$

Clearly, $r_1$ is a possible solution just when $a < b\left(2+\sqrt{3}\right)$. If we consider the condition that it should be a smooth tangent, then the circle $q_1$ should tangent the inside of a circle $q_2$, where $q_2$ is the circle, which approximates the ellipse in the co-vertex $C$.



(a)



(b)

Fig. 10

It follows from this condition that $|S_1S_2| = k + b - r_1$ where $|S_1S_2| = 2h$ and we have $k = (a-b)\left(3+\sqrt{3}\right)/2$. The second solution $\bar{k}_1$ can also be excluded because in this case the co-vertex $C$ of the ellipse would be the inside point of the circle $q_1$. When we know lengths of the major and the minor axes of the ellipse, for example, point $S_1$ must be constructed. If we express the radius as $r_1 = \left(a + b - \sqrt{3}(a-b)\right)/2$, we can construct a line

segment of length $\sqrt{3}(a-b)$ as in Fig. 10 (a), where $|AL| = a-b$, $|AL_1| = 1$, $|M\,L_1| = 2$ and $|AJ| = \sqrt{3}(a-b)$.
We presented in Fig. 10 (b) the construction of the ellipse whose $a / b = 5/4$ (red colour). The point $S_1$ is the center of the line segment $BB_1$, where $|LB_1| = |AJ| = \sqrt{3}(a-b)$, $|LB| = a+b$, $|AL| = a-b = 1$.

Guarini introduced the constructions of ovals which approximate the ellipse with known major axis. We modify his construction so that the oval approximates the ellipse even in its co-vertex. If we choose the radius $r_1$ and hence the center $F$ (Fig. 4 (b)), so $k = |ZH|$ is expressed in equation (2) and for the point $G$ must already apply $|FG| = |FH|$, so we cannot choose it arbitrarily.

We have the major and minor axes of the ellipse. Let the circle $q_1$ be the osculating circle for the point $A$ with the center at point $S_1$ (Fig. 11). For its radius $r_1 = b^2/a$ and the equation (2) shows that

$$|SS_2| = |SH| = k = \frac{(a-b)}{2}\frac{(a+2b)}{b}.$$

The point $S_2$ is the center of the circle $q_2$ which approximates the ellipse in the co-vertex $C$. Construction of line segment $SH$ is based on the similarity of the triangles $\Delta\, SHG_1$, $\Delta\, SFC$, $\Delta\, SKC$, where $|SG_1| = (a-b)/2$, $|SE| = 1$, $|SK| = a+2b$, $|SF| = (a+2b)/b$.

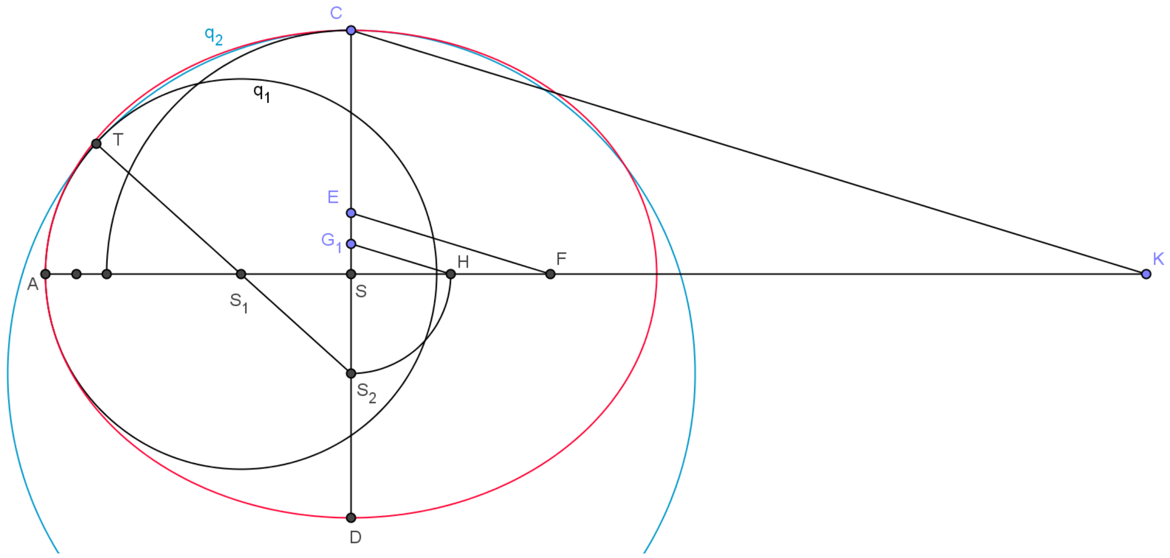



Fig. 11

We now compare our modified Guarini's construction with Meyer´s construction. It is clear that Meyer's construction is more precise (Fig. 12 and Fig. 13), because we use both osculating circles of the ellipse (green colour). In modified Guarini's construction we use the circle $q_2$ (blue colour in Fig. 12 or Fig. 13) which is not the osculating circle of the ellipse (red colour).

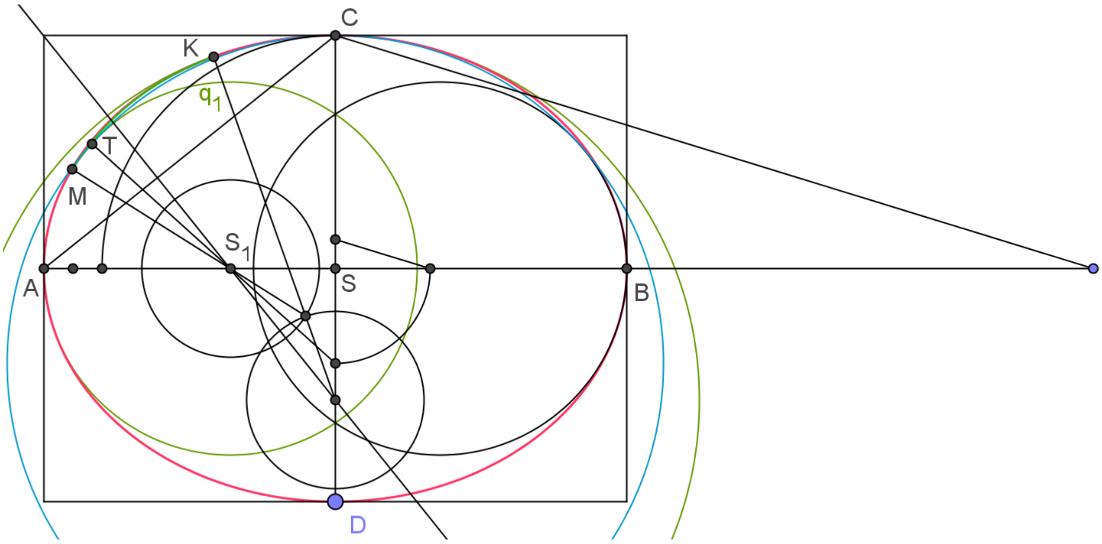



Fig. 12

Point *T* is the tangent point of the circles in the Guarini´s construction (Fig. 13), and points *M*, *K* are tangent points of the osculating arcs with a third circular arc in the Meyer´s construction.

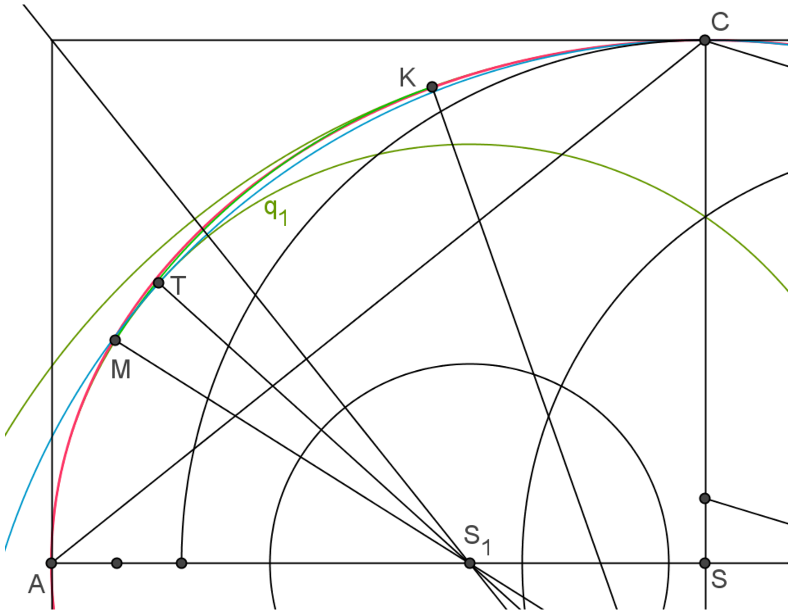



Fig. 13

We can say that given ellipses are the most precisely approximated by the Meyer´s constructions because he used osculating circles. Not even the modified Serlio´s and Guarini´s constructions do not give us more precise approximation of the ellipse than the Meyer´s construction in Fig. 9. However, we used only two circular arcs. The difference between the ovals presented in this article and the ellipse is minimal. Therefore, when we reconstruct some of buildings, we think that it is very difficult to determine unambiguously whether the ellipse was constructed or an oval that approximates this ellipse. Due to technical practice, we think that they were used ovals for approximation of the ellipse. For interpreting an elliptical shape, these constructions are, in our opinion, sufficient and practically usable. We believe they will continue to be used as they did in the past.

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