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REALISING SOME TYPES WHILE OMITTING OTHERS

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Abstract. We prove a theorem that combines omitting and realizing of types in one structure. Given two collections of types, we formulate a condition ensuring existence of a model realizing types from the first collection while omitting types from the second.

Keywords: saturated models, omitting types, realizing types

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1 Preliminaries

Let us first recall some important notions and fix notation that will be used throughout the paper.

1.1 Basic notation

By κ we denote infinite cardinal numbers; ω, resp. ω_1, denote the smallest infinite, resp. the smallest uncountable, cardinal number.

In the next, unless explicitly stated otherwise, $\mathcal{L}$ denotes a countable language, $\mathcal{M}$ a structure for $\mathcal{L}$, M denotes the universe of $\mathcal{M}$, and T a theory in $\mathcal{L}$.

If $A \subseteq M$, then $\mathcal{M}_A$ denotes the expansion of $\mathcal{M}$ by adding names for elements from A as new constants.

$\mathrm{Th}(\mathcal{M})$ denotes the set of all $\mathcal{L}$-formulas true in $\mathcal{M}$. $\mathrm{Th}(\mathcal{M}_M)$ is called the elementary diagram of $\mathcal{M}$ and is denoted by $\Delta_e(\mathcal{M})$.

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1.2 Types in theories, types in structures

By a type in T we mean a set $p(x)$ of $\mathcal{L}$-formulas in the free variable x, such that $p(x)$ is consistent with T (i.e. for any $\varphi_1(x), \ldots, \varphi_n(x) \in p(x)$, the theory $T \cup \{\exists x(\varphi_1(x) \wedge \cdots \wedge \varphi_n(x))\}$ is a consistent). We say that $p(x)$ is isolated if there exists an $\mathcal{L}$-formula $\psi(x)$ consistent with T and such that $\psi(x) \to \varphi(x)$ is provable in T for any $\varphi(x) \in p(x)$.

By a type over A in $\mathcal{M}$ we mean a type in the theory $\mathrm{Th}(\mathcal{M}_A)$.

1.3 Realizing and omitting

We say that $p(x)$ is realized in $\mathcal{M}$ by an element $m \in M$ if $\varphi(m)$ holds true in $\mathcal{M}$ for every $\varphi(x) \in p(x)$. If $p(x)$ is not realized by any $m \in M$, we say that $p(x)$ is omitted in $\mathcal{M}$.

A structure $\mathcal{M}$ is said to be saturated if for any $A \subseteq M$, such that $|A| < |M|$, every type in $\mathcal{M}$ over A is realized in $\mathcal{M}$ by some $m \in M$.

2 Motivation

Given a theory T, one might ask how its models differ from one another with respect to the types realized or omitted in them. The next two theorems are well-known results that describe two opposite approaches to this question. They can be found in [1] as Theorem 4.2.4 and Corollary 4.3.14.

In a complete theory, all the isolated types are always realized. Apart from that, there is no other restriction on what types can be omitted, as expressed in the Omitting Types Theorem.

Theorem 1 (Omitting Types Theorem)**.** *Let $\mathcal{L}$ be a countable language, T a consistent theory in the language $\mathcal{L}$. Let Ω be a countable collection of nonisolated types in T. Then there exists a countable model $\mathcal{M}$ of T omitting all the types from Ω.*

On the other hand, we can find a model realizing all the types we could wish for, under some assumptions.

Theorem 2. *Let $\mathcal{L}$ be a countable language, T a complete theory in $\mathcal{L}$ with infinite models. Suppose that $2^\kappa = \kappa^+$. Then there exists a saturated model of T of size κ^+.*

In our main theorem, we assume that the continuum hypothesis (CH) holds true, i. e. $2^\omega = \omega_1$. Therefore, in our case, the previous theorem ensures the existence of saturated structures of cardinality ω_1.

Of course there may be, in general, other models “in between” those two cases, i. e. models with some nonisolated types realized and others omitted. We concern ourselves with a situation when we are given two collections of types—collection Ω of types to be omitted and collection Σ of types to be realized. Our theorem is therefore filling in the gap between the theorem on existence of saturated models and the Omitting Types Theorem.

The motivation for our theorem comes not only from a theoretical interest, but from the study of real closed fields and their substructures, especially the so called integer parts, as presented in [2]. In particular, it is possible to study how automorphisms of real closed fields act on the substructures in question. A natural connection between automorphisms of a structure and saturation of its substructures is given by the following proposition.

Proposition 3. *If $\mathcal{A}$ is a structure and a its automorphism, then any saturated substructure of $\mathcal{A}$ is mapped by a onto a saturated substructure of $\mathcal{A}$.*

In our specific case, we were led to look for substructures of real closed fields that are saturated in the language containing only the order relation $\leq$, but not saturated when the whole language of arithmetic is considered. We believe that this is an example of the most natural application of our theorem—the collections Σ, of types to be realized, and Ω, of types to be omitted, are distinguished by the language; Σ contains types only in certain sublanguage, while the types from Ω can be in the full language in question.

3 Main result

Before we proceed to a formulation of our main theorem, we need one more notion. As we have indicated above, our aim is to find a structure $\mathcal{M}$ realizing all the types from the collection Σ. We would like to allow parameters from M in those types. For this reason, we allow countably many parametric variables v_i to appear in the formulas.

Definition 4. *A countably parametrized set of $\mathcal{L}$-formulas in the variable x is a set of $\mathcal{L}$-formulas whose all free variables belong to $\{x\} \cup \{v_i; i \in \omega\}$.*

In the next, we denote such countably parametrized sets of formulas by $s(x, \bar{v})$. Once we have obtained some structure $\mathcal{M}$ and are given a sequence of its elements $\bar{m} \in \mathcal{M}^\omega$, it is possible to "substitute" $\bar{m}$ for $\bar{v}$ and hence understand $s(x, \bar{m})$ as a set of formulas with parameters from M. Such an $s(x, \bar{m})$ may, or may not, become a type in $\mathcal{M}$, depending on the choice of $\bar{m}$.

If Σ is a collection of such countably parametrized sets of formulas and $e : \omega_1 \to \Sigma$ is its enumeration, we write $s_\alpha(x, \bar{v})$ for $e(\alpha)$.

Theorem 5. *(CH) Let $\mathcal{L}$ be a countable language, T a consistent theory. Let Ω be a countable collection of nonisolated types in T and let Σ be a collection of countably parametrized sets of $\mathcal{L}$-formulas. Furthermore, let us fix an enumeration of Σ.*

Suppose that for any $\beta < \omega_1$ and any countable model $\mathcal{M}$ of T omitting all types from Ω, it holds that no $p(x) \in \Omega$ is isolated in the theory

$$\bigcup \{s_\alpha(c_{\alpha,\bar{m}}, \bar{m}); \alpha \leq \beta, \bar{m} \in M^\omega, s_\alpha(x, \bar{m}) \text{ is a type of } \mathcal{M}\} \cup \Delta_e(\mathcal{M}),$$

where $\{c_{\alpha,\bar{m}}; \alpha \leq \beta, \bar{m} \in M^\omega\}$ is a set of fresh constants.

Then there exists a model $\mathcal{M}$ of T omitting every $p(x) \in \Omega$ and realizing every type in $\mathcal{M}$ of the form $s(x, \bar{m})$, where $s(x, \bar{v}) \in \Sigma$ and $\bar{m} \in M^\omega$.

Proof. First, we fix a set U of cardinality ω_1 and an enumeration $\{\bar{u}_\gamma\}_{\gamma \in \omega_1}$ of U^ω. The universe of the model $\mathcal{M}$ we look for will be a subset of U.

We construct an elementary chain $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \ldots \preccurlyeq \mathcal{M}_\beta \preccurlyeq \ldots$ of length ω_1 satisfying the following conditions:

1. For any $\beta < \omega_1$, $\mathcal{M}_\beta$ is a countable model of T omitting all types from the collection Ω, $M_\beta \subseteq U$.
2. Let $\beta < \omega_1$, and let $\alpha, \gamma < \beta$. If there is $\delta < \beta$ such that $\bar{u}_\gamma \in (M_\delta)^\omega$ and $s_\alpha(x, \bar{u}_\gamma)$ is a type of $\mathcal{M}_\delta$, then it is realized in $\mathcal{M}_\beta$.

By the Omitting Types Theorem, there exists a countable model $\mathcal{M}_0$ of T omitting all types from Ω. Moreover, it can be chosen so that $M_0 \subseteq U$. Such an $\mathcal{M}_0$ clearly satisfies the conditions above.

Suppose λ is a limit ordinal and we have already constructed the elementary chain $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \ldots \preccurlyeq \mathcal{M}_\beta \preccurlyeq \ldots$ of length λ satisfying all the above conditions up to λ. If we extend this elementary chain by adding $\mathcal{M}_\lambda := \bigcup_{\beta<\lambda} \mathcal{M}_\beta$ as its last element, it can be easily checked that such an extension satisfies the above condition up to and including λ.

Suppose we have already constructed the chain up to $\mathcal{M}_\beta$; we show how to obtain $\mathcal{M}_{\beta+1}$. Consider a theory

$$\bigcup \{s_\alpha(c_{\alpha,\bar{u}_\gamma}, \bar{u}_\gamma); \alpha, \gamma \leq \beta, \bar{u}_\gamma \in (M_\beta)^\omega, s_\alpha(x, \bar{u}_\gamma) \text{ is a type of } \mathcal{M}_\beta\} \cup \Delta_e(\mathcal{M}_\beta),$$

where $\{c_{\alpha,\bar{u}_\gamma}; \alpha, \gamma \leq \beta\}$ is a set of fresh constants.

It is a consistent theory in a countable language and, by the assumption of the theorem, no $p(x) \in \Omega$ is isolated in it. Hence, using again the Omitting Types Theorem, it has a countable model $\mathcal{M}_{\beta+1}$ omitting all types from Ω; moreover such that $M_{\beta+1} \subseteq U$. It is obvious that $\mathcal{M}_{\beta+1}$ satisfies condition 1. Condition 2 is satisfied as well, since the theory makes sure that the type $s_\alpha(x, \bar{u}_\gamma)$ is satisfied by the newly added constant $c_{\alpha,\bar{u}_\gamma}$.

Finally, let us define $\mathcal{M} := \bigcup_{\beta<\omega_1} \mathcal{M}_\beta$. It is a union of an elementary chain of models of T and therefore also a model of T. It ommits all types from Ω as all $\mathcal{M}_\beta$ do so. Let $\bar{m} \in M^\omega$, $\alpha < \omega_1$ and suppose $s_\alpha(x, \bar{m})$ is a type in $\mathcal{M}$. Then, by regularity of ω_1, there is some $\beta < \omega_1$ such that $\bar{m} \in (M_\beta)^\omega$. Put $\mu := max(\alpha, \beta)$; since $\mathcal{M}_\mu \preccurlyeq \mathcal{M}$, it holds that $s_\alpha(x, \bar{m})$ is also a type in $\mathcal{M}_\mu$. By condition 2, it is realized in $\mathcal{M}_{\mu+1}$ and therefore also in $\mathcal{M}$.

□

4 Application

We present one application of our main theorem as an illustration of its use. In this section, $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{P}$ denotes the set of all prime numbers.

Let Pr be the additive theory of the structure of natural numbers and $p(x)$ a type expressing that x is a nonzero element divisible by all primes; i. e. $\mathrm{Pr} = \mathrm{Th}(\langle \mathbb{N}, 0, 1, +, \leq \rangle)$ and $p(x)$ is the set $\{x \neq 0\} \cup \{p|x; p \in \mathbb{P}\}$.[1] We show how to obtain a model of Pr saturated in the language $\{0, 1, \leq\}$ omitting $p(x)$.

Lemma 6. *Let $\mathcal{M}$ be model of* Pr *and let $s(x)$ be a type in $\mathcal{M}$ in the language $\{0, 1, \leq\}$ omitted in $\mathcal{M}$. Let us fix a function $\tau : \mathbb{P} \to \mathbb{N}$, such that $\tau(p) \in \{0, 1, \ldots, p-1\}$. Then $s'(x) = s(x) \cup \{x \bmod p = \tau(p); p \in \mathbb{P}\}$ is also a type in $\mathcal{M}$.*

Proof. Let us fix $\mathcal{M}'$, a model of $\Delta_e(\mathcal{M})$ realizing $s(x)$ by some some element $m \in M'$ (such a model exists by Theorem 2). Then the $\mathbb{Z}$-component of m, i.e. the set $Z = \{m' \in M'; |m - m'| \in \mathbb{N}\}$, is disjoint from M (else m would be an element of $\mathcal{M}$). Therefore all the parameters from $s(x)$ lie outside of Z. By considering all the order-automorphisms of $\mathcal{M}'$ shifting the elements of Z while fixing all the other, in particular all parameters from $s(x)$, it is easy to see that any element of Z realizes $s(x)$. By the Chinese remainder theorem, every finite part of $s'(x)$ is realized by some element of Z. Therefore $s'(x)$ is a type. □

The lemma above tells us that we can control moduli of elements realizing types in the language $\{0, 1, \leq\}$. This allows us to avoid the type $p(x)$ from being realized, as stated in the next lemma.

[1] The formula $p|x$ can be expressed in the additive language as $(\exists y)(\underbrace{y + y + \cdots + y}_{p\text{-times}} = x)$.

Lemma 7. *Let $\mathcal{M}$ be a countable model of* Pr *omitting* $p(x)$. *Let* Σ *be a countable family of types in the language* $\{0, 1, \leq\}$ *in* $\mathcal{M}$. *Then there exists a model* $\mathcal{M}'$ *of* $\Delta_e(\mathcal{M}) \cup \{s(c_s); s \in \Sigma\}$, *where* $\{c_s, s \in \Sigma\}$ *is a set of fresh constants, omitting* $p(x)$.

If T is the theory Pr, $\Omega = \{p(x)\}$, and Σ is the set of all countably parametrized sets of formulas in the language $\{0, 1, \leq\}$, then, by Lemma 7, the assumption of Theorem 5 is satisfied. Therefore, as an application, we get the following result.

Theorem 8. *(CH) There exists a model* $\mathcal{M}$ *of* Pr *saturated in the language* $\{0, 1, \leq\}$ *and omitting the type* $\{x \neq 0\} \cup \{p|x; p \in \mathbb{P}\}$.

5 Further research

There are two basic areas that offer opportunity for further research.

First, one could try to get rid of the assumption of the continuum hypothesis and find a natural reformulation of our theorem with no additional set theoretic assumptions.

Second, one could try to come up with more algebraic applications of our theorem, apart from the application in the study of real closed fields and their integer parts we have already mentioned.

References

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