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FINANCIALLY DEPENDENT PENSIONS FUNDS MAINTENANCE APPROACH THROUGH BROWNIAN MOTION PROCESSES

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Abstract. The situation of some pensions funds that are not appropriately auto financed and are thoroughly maintained with an outside financing effort is considered in this paper. To represent the unrestricted reserves value process of this kind of funds, a time homogeneous diffusion stochastic process is proposed. Then it is projected a financial tool that regenerates the diffusion at some level with positive value. So, the financing effort may be modeled as a renewal-reward process. The relevant costs are studied when the unrestricted reserves value process behaves as a generalized Brownian motion process.

Keywords: pensions fund, diffusion process, Brownian motion process, first passage time, perpetuity, renewal equation

Mathematics Subject Classification: 60J70

1 Introduction

The sustainability of a pensions fund study is performed in [4, 6 and 11], in the scope of queuing theory. Two infinite servers' queues are considered: one with the contributors to the fund, which service time is the time during which they pay to the fund; the other with the pensioners which service time is the time during which they receive the pension. In both queues there is no distinction between the customer and its server in queuing terms technical sense.

The funds portrayed herein are the so-called distributive pension's funds. State pension's funds are generally of this kind. Their income is the contributions of the employees that are distributed by pensioners in the form of retirement pensions. The most important result of these studies is that, in order for the fund to be in balance, the average pension should be equal - or lesser than - to the average contribution. However, due to the demographic imbalance that is happening in contemporary societies, with the number of taxpayers consecutively decreasing while the number of pensioners increases inexorably, it has meant that if the funds were to be autonomously balanced the contributions of the workers would have to assume insupportable values.

What usually is happening is the injection of capital into these funds through transfers from the State Budget, whenever necessary. Therefore in a disorganized way and, in general, in unforeseen situations that may coincide with large financial difficulties. This study purpose is to try to make these situations more predictable, both in relation to the time of occurrence and the amount needed, so that the protection of these funds occurs so smoothly as possible.

Along this paper, that problem is approached bearing in mind the protection cost present value expectation for a non-autonomous pension's fund. Two problems are considered in this context:
-*The above mentioned expectation when the protection effort is perpetual*,
-*The protection effort for a finite time period*.
It is admitted that the unrestricted fund reserves behavior may be modeled as a time homogeneous diffusion process and use then a regeneration scheme of the diffusion to include the effect of an external financing effort.

A similar work may be seen in [15], where has been considered a Brownian motion process conditioned by a particular reflection scheme. Less constrained, but in different conditions, exact solutions were then obtained for both problems.

Part of this work was presented at the Fifth International Congress on Insurance: Mathematics & Economics, see [12]. Other works on this subject are [3, 7, 9 and 10].

2 Diffusion Processes and Pensions Funds

Be $X(t), t \geq 0$ the reserves value process of a pensions fund given by an initial reserve amount $a, a > 0$ added to the difference between the total amount of contributions received up to time *t* and the total amount of pensions paid up to time *t*. It is assumed that $X(t)$ is a time homogeneous diffusion process, with $X(0) = a$, defined by drift and diffusion coefficients:

$$\lim_{h \to 0} \frac{1}{h} E[X(t+h) - X(t) | X(t) = x] = \mu(x),$$

$$\lim_{h \to 0} \frac{1}{h} E\left[\left(X(t+h) - X(t)\right)^2 | X(t) = x\right] = \sigma^2(x).$$

Call S_a the first passage time of $X(t)$ by 0, coming from a. The funds to be considered in this work are non-autonomous funds. So

$$E[S_a] < \infty, \text{for any } a > 0 \qquad (2.1),$$

that is: funds where the pensions paid consume in finite expected time any initial positive reserve and the contributions received, so that other financing resources are needed in order that the fund survives.

The condition (2.1) may be fulfilled for a specific diffusion process using criteria based on the drift and diffusion coefficients. Begin stating that $P(S_a < \infty) = 1$ if the diffusion scale function is

$$q(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

where x_0 is a diffusion state space fixed arbitrary point, fulfilling $q(\infty) = \infty$. Then the condition (2.1) is equivalent to $p(\infty) < \infty$, where

$$p(x) = \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

is the diffusion speed function. It is thought that, whenever the exhaustion of the reserves happens, an external source places instantaneously an amount $\theta, \theta > 0$ of money in the fund so that it may keep on performing its function.

The reserves value process conditioned by this financing scheme is represented by the modification $\check{X}(t)$ of $X(t)$ that restarts at the level θ whenever it hits 0. Note that since $X(t)$ was defined as a time homogeneous diffusion, $\check{X}(t)$ is a regenerative process. Call $T_1, T_2, T_3, \ldots$ the sequence of random variables where T_n denotes the n^{th} $\check{X}(t)$ passage time by 0. It is obvious that the sequence of time intervals between these hitting times $D_1 = T_1, D_2 = T_2 - T_1, D_3 = T_3 - T_2, \ldots$ is a sequence of independent random variables where D_1 has the same probability distribution as S_a and $D_2, D_3, \ldots$ the same probability distribution as S_θ.

3 First passage times Laplace transforms

Call $f_a(s)$ the probability density function of $S_a (D_1)$. The corresponding probability distribution function is denoted by $F_a(s)$. The Laplace transform of S_a is

$$\varphi_a(\lambda) = E\left[e^{-\lambda S_a}\right] = \int_0^{\infty} e^{-\lambda s} f_a(s) ds, \lambda > 0.$$

Consequently, the density, distribution and transform of S_θ $(D_2, D_3, \ldots)$ will be denoted by $f_\theta(s), F_\theta(s)$ and $\varphi_\theta(\lambda)$, respectively.

The transform $\varphi_a(\lambda)$ satisfies the second order differential equation

$$\frac{1}{2}\sigma^2(a)u_\lambda''(a) + \mu(a)u_\lambda'(a) = \lambda u_\lambda(a), u_\lambda(a) = \varphi_a(\lambda), u_\lambda(0)=1, u_\lambda(\infty) = 0 \quad (3.1).$$

For details see [7].

4 Perpetual Maintenance Cost Present Value

Consider the perpetual maintenance cost present value of the pension's fund that is given by the random variable

$$V(r, a, \theta) = \sum_{n=1}^{\infty} \theta e^{-rT_n}, \text{ r>0},$$

where *r* represents the appropriate discount rate. Note that $V(r, a, \theta)$ is a random perpetuity: what matters is its expected value which is easy to get using Laplace transforms. Since the T_n Laplace transform is $E\left[e^{-\lambda T_n}\right] = \varphi_a(\lambda)\varphi_\theta^{n-1}(\lambda)$,

$$v_r(a, \theta) = E[V(r, a, \theta)] = \frac{\theta\varphi_a(r)}{1-\varphi_\theta(r)} \quad (4.1).$$

It is relevant to note[1] that

[1] See again [7] for details.

$$\lim_{\theta \to 0} v_r(a,\theta) = \frac{u_r(a)}{-u_r^{'}(0)} \qquad (4.2).$$

5 Brownian Motion Process

Consider that the diffusion process $X(t)$ underlying the reserves value behavior of the pensions fund is a generalized Brownian motion process, with drift $\mu(x) = \mu, \mu < 0$ and diffusion coefficient $\sigma^2(x) = \sigma^2, \sigma > 0$. Observe that the setting satisfies the conditions that were assumed initially. Namely $\mu < 0$ point toward condition (2.1). Everything else remaining as previously stated, it will be proceeded to present the consequences of this particularization. In general it will be added a symbol $(*)$ to the notation used before because it is intended to use these specific results later.

To get the first passage time S_a Laplace transform it must be solved, remember (3.1),

$$\frac{1}{2}\sigma^2(a)u_\lambda^{*''}(a) + \mu(a)u_\lambda^{*'}(a) = \lambda u_\lambda^*(a), u_\lambda^*(a) = \varphi_a(\lambda), u_\lambda^*(0)=1, u_\lambda^*(\infty) = 0 .$$

This is a homogeneous second order differential equation with constant coefficients, which general solution is given by

$$u_\lambda^*(a) = \beta_1 e^{\alpha_1 a} + \beta_2 e^{\alpha_2 a}, with\ \alpha_1, \alpha_2 = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}.$$

Condition $u_\lambda^*(\infty) = 0$ implies $\beta_1 = 0$ and $u_\lambda^*(0)=1$ implies $\beta_2=1$ so that the particular solution is achieved:

$$u_\lambda^*(a) = e^{-K_\lambda a}\left(= \varphi_a^*(\lambda)\right), K_\lambda = \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \qquad (5.1).$$

In this case, the perpetual maintenance cost present value of the pensions fund is given by, following (4.1) and using (5.1),

$$v_r^*(a,\theta) = \frac{\theta e^{-K_r a}}{1 - e^{-K_r \theta}} \qquad (5.2).$$

Note that $v_r^*(a,\theta)$ is a decreasing function of *a* and an increasing function of θ. In particular

$$\lim_{\theta \to 0} v_r^*(a,\theta) = \frac{e^{-K_r a}}{K_r} \qquad (5.3).$$

To reach an expression for the finite time period maintenance cost present value, start by the computation of $k_r^*(\theta)$, finding a positive *k* satisfying $e^{-K_r\theta}e^{-K_{-\lambda}\theta} = 1$ or $K_r + K_{-\lambda} = 0$. This identity is verified for the value of *k*

$$k_r^*(\theta) = \frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \qquad (5.4).$$

Note that the solution is independent of θ in these circumstances. An easy solution, independent of *a* and θ , for $c_r^*(a,\theta)$ was also obtained:

$$c_r^*(a,\theta) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2} \qquad (5.5).$$

So it is observable that the asymptotic approximation for this particularization reduces to $w_r^*(t; a, \theta) \approx v_r^*(a, \theta) - \pi_r(t)$, where the function $\pi_r(t)$ is, considering (5.4) and (5.5),

$$\pi_r(t) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}\, e^{-\frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}t}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \qquad (5.6).$$

6 Representation of the Assets and Liability Behaviour

Now an application of the results obtained earlier to an asset-liability management scheme of a pensions fund will be presented. Assume that the assets value process of a pensions fund may be represented by the geometric Brownian motion process

$$A(t) = be^{a + (\rho + \mu)t + \sigma B(t)} \text{ with } \mu < 0 \text{ and } ab\rho + \mu\sigma > 0,$$

where $B(t)$ is a standard Brownian motion process. Suppose also that the liabilities value process of the fund performs as the deterministic process $L(t) = be^{\rho t}$. Under these assumptions, consider now the stochastic process $Y(t)$ obtained by the elementary transformation of $A(t)$

$$Y(t) = ln\frac{A(t)}{L(t)} = a + \mu t + \sigma B(t).$$

This is a generalized Brownian motion process exactly as the one studied before, starting at a and with drift μ and diffusion coefficient σ^2. Note also that the firs passage time of the assets process $A(t)$ by the mobile barrier T_n, the liabilities process, is the first passage time of $Y(t)$ by 0-with finite expected time under the condition, stated before, $\mu < 0$.

Consider also the pensions fund management scheme that raises the assets value by some positive constant θ_n when the assets value falls equal to the liabilities process by the n^{th} time. This corresponds to consider the modification $\bar{A}(t)$ of the process $A(t)$ that restarts at times T_n when $A(t)$ hits the barrier $L(t)$ by the n^{th} time at the level $L(T_n) + \theta_n$. For purposes of later computations it is a convenient choice the management policy where

$$\theta_n = L(T_n)\left(e^\theta - 1\right), \text{for some } \theta > 0 \quad (6.1).$$

The corresponding modification $\tilde{Y}(t)$ of $Y(t)$ will behave as a generalized Brownian motion process that restarts at the level $ln\frac{L(T_n)+\theta_n}{L(T_n)} = \theta$ when it hits 0 (at times T_n).

Proceeding this way, it is reproduced via $\tilde{Y}(t)$ the situation observed before when the Brownian motion example was treated. In particular the Laplace transform in (5.1) is still valid.
Similarly to former proceedings, the results for the present case will be distinguished with the symbol $(\#)$. It is considered the pensions fund perpetual maintenance cost present value, as a consequence of the proposed asset-liability management scheme, given by the random variable:

$$V^{\#}(r,a,\theta)=\sum_{n=1}^{\infty}\theta_n e^{-rT_n}=\sum_{n=1}^{\infty}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n},r>\rho$$

where r represents the appropriate discount interest rate. To obtain the above expression it was only made use of the $L(t)$ definition and (6.1). It is possible to express the expected value of the above random variable with the help of (5.2) as

$$v_r^{\#}(a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}v_{r-\rho}^{*}(a,\theta)=\frac{b\left(e^{\theta}-1\right)e^{-K_{r-\rho}a}}{1-e^{-K_{r-\rho}\theta}} \qquad (6.2).$$

As $\theta\to 0$

$$\lim_{\theta\to 0}v_r^{\#}(a,\theta)=\frac{be^{-K_{r-\rho}a}}{K_{r-\rho}} \quad (6.3).$$

In a similar way, the maintenance cost up to time t in the above mentioned management scheme, is the stochastic process

$$W^{\#}(t;r,a,\theta)=\sum_{n=1}^{N(t)}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n},\ W^{\#}(t;r,a,\theta)=0 \text{ if } N(t)=0,$$

with expected value function

$$w_r^{\#}(t;a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}w_{r-\rho}^{*}(t;a,\theta) \qquad (6.4).$$

7 Conclusions

The whole work established depends on the possibility of solving equation (3.1) to obtain the first passage times Laplace transforms. Unfortunately, the solutions are known only for rare cases. An obvious case for which the solution of the equation is available is the one of the Brownian motion diffusion process. The main results concerning this particularization are formulae (5.2) and (5.6). Certain transformations of the Brownian motion process that allowed use of the available Laplace transform may be explored as it was done in section 6. Formulae (6.2) and (6.4) are this application most relevant results.

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