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CONSTRUCTIONS OF A SQUARE AND A REGULAR PENTAGON ONLY WITH A COMPASS

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Abstract. In 1797, Lorenzo Mascheroni proved that a compass used for constructions has the equal power as using a ruler altogether with the compass. Therefore, it is possible to construct all ruler-and-compass constructions only with the compass itself. The aim of this article is not to prove the theorem but to introduce interesting constructions of a square and a regular pentagon.

Keywords: Mascheroni construction, square, regular pentagon

Mathematics subject classification: Primary 51N05; Secondary 01A50, 97G80

1 Introduction

Constructional and descriptive geometry are integral parts of university studies of the future Maths teachers, engineers, constructors and architects. Students must be able to interconnect and apply geometric knowledge and use their imagination. [6] The ruler-and-compass constructions (sometimes called Euclidean constructions) and their slight modifications are taught as a part of constructional geometry. However, there are other methods equal to Euclidean constructions, using different means. One of them are Mascheroni constructions which allow only the use of a compass. Lorenzo Mascheroni was an important Italian mathematician and even Napoleon himself was impressed with Mascheroni's work at that time. The most important part of his work is the so-called Mohr-Mascheroni Theorem, stating that every geometric construction carried out by a compass and a ruler can be done without a ruler. An indication of the theorem can be found in Euclidia Danubis, a book by Georg Mohr. Another proof based on a circle inversion, was created by Adler [1] or Kostovskii [3].

2 Georg Mohr and Lorenzo Mascheroni

Georg Mohr (Jorgen Mohr in Danish) was a mathematician, born in Copenhagen in 1640. Because there is a lack of information about his life, it is assumed he left Denmark to the Netherlands to study one of their universities as they were of high prestige at that time. His book Euclides Danicus, published in 1672, described the main idea of Mascheroni's theory in Danish, thus there was a belief it is only a translation of Euclid's work Elements. It was only in 1927 when professor Johan Hjemslevov discovered the importance of the work thanks to one his students who had found a copy of Euclides Danicus in a secondhand bookshop. [5] The second book of Mohr's, Compendium Euclides Curiosi, was published in 1673 and it is based on Elements again. Interestingly, a letter dated on May 12th, 1676 from Leibnitz to H. Oldenburg, a secretary of the Royal Society in London is still in existence, stating: "Georgius Mohr Danus, in geometria et analysi versatissimus". Leibnitz spend some time in Paris and it is possible he and Mohr met or Leibnitz read Mohr's work. [2] In 1681, Mohr travelled back to Denmark and applies for a job at Danish Royal Court. Although the king offered him supervision of building royal navy, he declined it. Mohr got married in 1687 and moved to the Netherlands supposedly due to a dispute with the king. During the last years of his life Mohr accepted Tschirnhaus's offer of job and moved with his family to Kieslingwald in Germany. [5] Nearly 200 years later, Lorenzo Mascheroni (1750-1800), Italian priest, poet and mathematician, was born in Bergamo. At his 17, he was ordained a priest and continued in his studies of rhetorics, which he began to teach in Bergamo 5 years later. Philosophy was his another field of interest - it included basics of logics and physics at that time. In 1786, he starts to work as a professor of algebra and geometry at university in Pavia. In the same year, he published a book called Nuove ricerche sull' equilibrio delle volte (1786) and later other important works, e.g. Adnotationes ad calculium integralem Euleri (1792), Problemi per gli agrimensori (1793) a Geometria del Compasso (1797). The latter work is the main one in our perspective as Mascheroni depicts Euclidean constructions which can be constructed only with compasses. [7]

Fig. 1. Lorenzo Mascheroni: La Geometria del Compasso [4]

La Geometria del Compasso is a complete work written in the Old Italian language by Mascheroni. The book is divided into 12 main chapters, each dealing with issues of the given topic. The structure of chapters is methodically arranged: a task, a description of a construction, a proof of construction, and an alternative solution (if there is one). The final constructions are to be found in the appendix of the book. Mascheroni used not only circles, but also arcs to achieve better arrangement of schemes. Each chapter moves from the easiest constructions to the most difficult ones. For instance, in a chapter, which focuses on division and lengthening of line segments, Mascheroni starts with a division of a line segment into 2 parts of the same length, then continues with a division into 4, 8 parts etc. Then he shows a division into an odd amount of parts and ends with construction of division of a line segment into n parts.

3 A square of a side a and an unknown diagonal u

Problem Let $AB,\ |AB| = a$ be a side of a square. Construct the square $ABCD$.

The problem can be solved in more ways. First of all, we must focus on a construction of two isosceles triangles ABC and ABD. To do that, we have to geometrically construct the side AC, which fuctions as the diagonal u of the square $ABCD$. If a side of the square is a then $|AC| = a\sqrt{2}$. Thus our goal is a construction of a line segment, i.e. a diameter of a circle of the length $a\sqrt{2}$.

Construction protocol

0. Points A, B; $|AB| = a$; circle k; $k(B, r = a)$
1. Point X; $X \in k \wedge |AX| = 2a$
   a) Circle $l_1; l_1(A, r = a)$
   b) Point $X_1; X_1 \in k \cap l_1$
   c) Circle $l_2; l_2(X_1, r = a)$
   d) Point $X_2; X_2 \in k \cap l_2$
   e) Circle $l_3; l_3(X_2, r = a)$
   f) Point $X; X \in k \cap l_3$
2. Circle $m_1; m_1(X, r = |XX_1|)$
3. Circle $m_2; m_2(A, r = |AX_2|$
4. Point $Y; Y \in m_1 \cap m_2$
5. Circle $n_1; n_1(B; r = |BY|)$
6. Point $D; D \in l_1 \cap n_1$
7. Circle $n_2; n_2(D; r = a)$
8. Point $C; C \in k \cap n_2$
9. Square $ABCD$

The line segment $a\sqrt{2}$ has occured because of the right-angled triangle ABY with side AB of length a, whereas BY is the value we have been seeking. However, the side AY of length $a\sqrt{3}$ is the essential one. But how the length $a\sqrt{3}$ was constructed? The circle $k(B, r = a)$ of radius a and center B is the so-called Thales' circle, i.e. any triangle ABQ, $Q \in k$, is right-angled. Let's focus on AXX_2, although we could discuss AXX_1 as well. The length of AX is $2a$ as AX represents doubled AB in the construction. XX_2 has the length a because the point X is the intersection of two

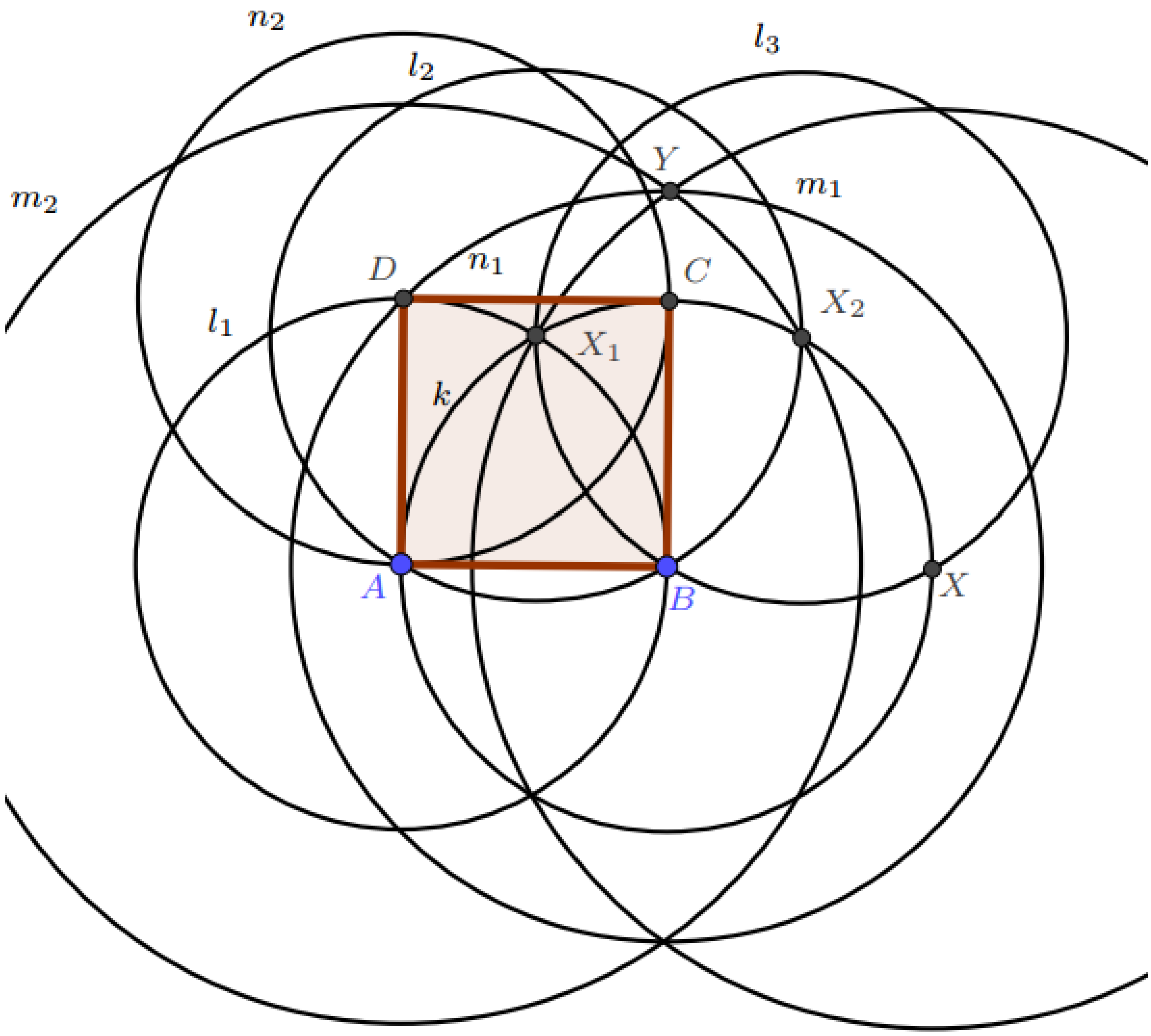

Fig. 2. The square $ABCD$ (A square of a side a and an unknown diagonal u)

circle, one of which is $l_3(X_2, r = a)$. The last side of the triangle is AX_2, whose side length can be calculated with Pythagorean theorem (with the right angle at X_2):

$$\begin{aligned} |AX|^2 &= |XX_2|^2 + |AX_2|^2 \\ (2a)^2 &= a^2 + |AX_2|^2 \\ 3a^2 &= |AX_2|^2 \\ a\sqrt{3} &= |AX_2| \end{aligned}$$

$|AX_2|$, and $|XX_1|$ as well, figures as the radius of the circles which created the intersection Y so that $|AX_2| = |XX_1| = |AY|$.

The second version of construction is based on the construction of a perpendicular to AB in the point A, on which the vertex D lies.

Construction protocol

0. Points A, B; $|AB| = a$; circle k; $k(B, r = a)$
1. Point $Y; AY \perp AB$
   a) Circle $l_1; l_1(A, r = a)$
   b) Point $S; S \in k \cap l_1$
   c) Circle $l_2; l_2(S, r = a)$
   d) Point $X; X \in l_1 \cap l_2$
   e) Circle $l_3; l_3(X, r = a)$
   f) Point $Y; Y \in l_2 \cap l_3$
2. Point $W; W \in l_1 \cap l_3$
3. Circle $m_1; m_1(W, r = |SW|)$
4. Circle $m_2; m_2(B, r = |BX|)$
5. Point $Z; Z \in m_1 \cap m_2$
6. Circle $n_1; n_1(W, r = |AZ|)$
7. Point D; $D \in l_1 \cap n_1$
8. Circle $n_2; n_2(C, r = a)$
9. Point $C; C \in k \cap n_2$
10. Square $ABCD$

This construction constists of several simpler Mascheroni constructions. First, the perpendicular AY is constructed, using the properties of Thales' circle l_2. Doubling the segment BS, the point Y was created and BY functions as the hypotenuse of the triangle ABY. Therefore there must be a right angle at A and $AY \perp AB$. Points X, S have occured during the construction. Both points lie on the circle k, forming the arch SX. As the radius of the circe l_1 equals a and the triangle ASX is isosceles with sides of enght a, the centre of the arch SX must be the vertex D of the square. Here we divide the arch using Mascheroni construction. The triangle ASX is transformed into two parallelograms $ABSX$ and $ASWX$. Advantageously, the construction continues just as the preceding one. If we doubled AB into AX and constructed the point Y in order to get BY as the diagonal $a\sqrt{2}$, now we double AW into BW and search the point Z that $AZ = a\sqrt{2}$. Both version of solving the problem eventually merge into one although each of them uses different Mascheroni constructions.

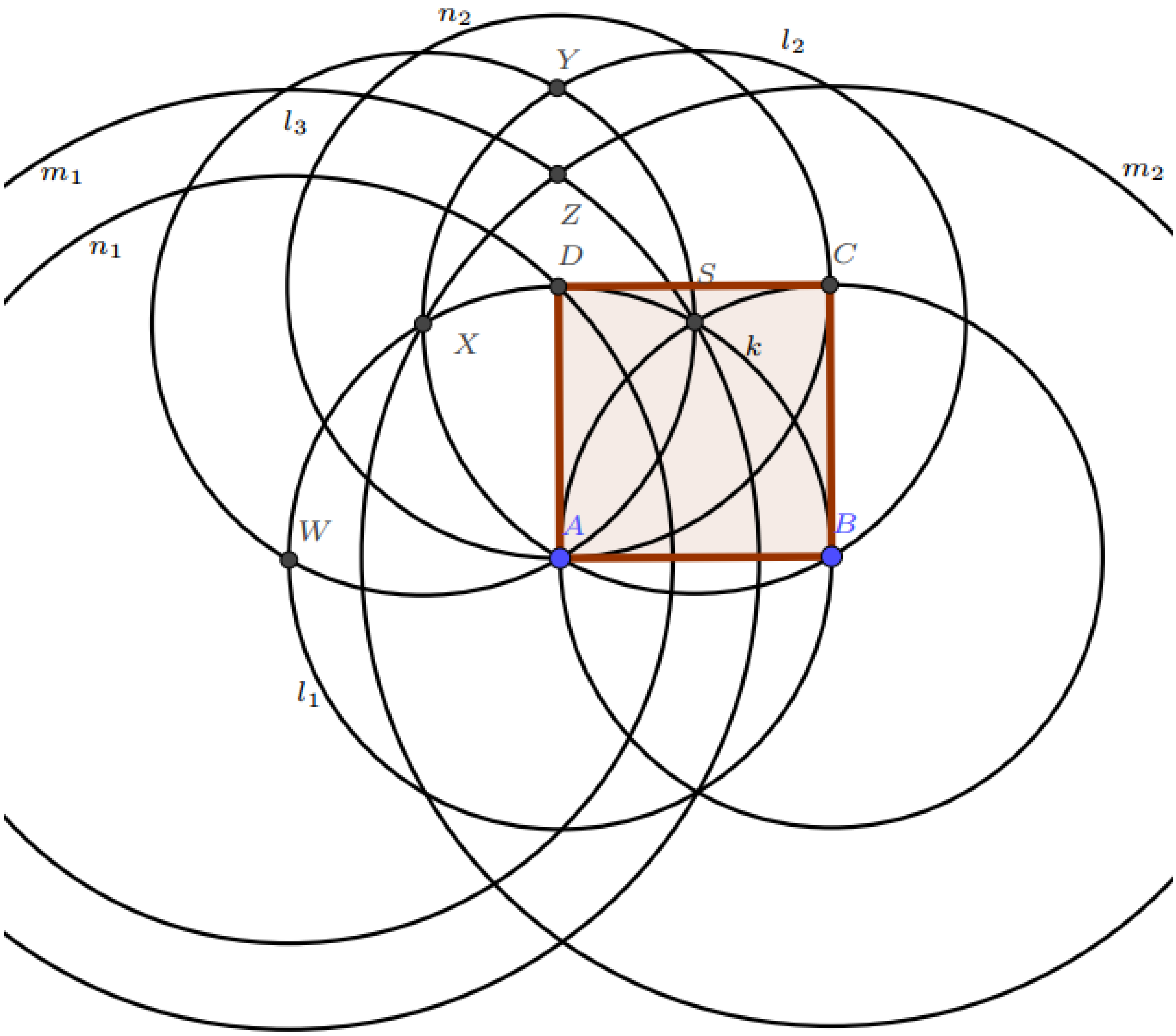

Fig. 3. The square $ABCD$ (A square of a side a and an unknown diagonal u)

4 A square of a diagonal u and an unknown side a

Problem Let AC, $|AC| = u$ be a diagonal of a square. Construct the square $ABCD$ with the diagonal AC.

Since we know the length $u = a\sqrt{2}$, we can change the problem to a construction of a square whose circumscribed circle has the radius $\frac{u}{2} = \frac{a\sqrt{2}}{2}$.

Construction protocol

0. Point S; circle k; $k(S, r = \frac{u}{2})$; bod A; $A \in k$
1. Point C; $C \in k \wedge |AC| = 2|AS|$
   a) Circle l_1; $l_1(A, r = \frac{u}{2})$

b) Point $X_1; X_1 \in k \cap l_1$

c) Circle $l_2; l_2(X_1, r = \frac{u}{2})$

d) Point $X_2; X_2 \in k \cap l_2$

e) Circle $l_3; l_3(X_2, r = \frac{u}{2})$

f) Point $C; C \in k \cap l_3$

2. Circle $m_1; m_1(A, r = |AX_2|)$
3. Circle $m_2; m_2(C, r = |CX_1|)$
4. Point $Y; Y \in m_1 \cap m_2$
5. Circle $n; n(A, r = |SY|)$
6. Points B, D; $B, D \in n \cap k$
7. Square $ABCD$

As in the first case, the construction includes doubling a line segment, Thales' circle k and using the knowledge of Pythagorean theorem. Regarding the difficulty, it is easier than first case because whether seeking the point B as the center of the arch X_1X_2 or the intersection of circles of radii $\frac{u}{2}$ and a, we always get the point D, too.

5 A construction of a regular pentagon of a side a.

Problem Let AB, $|AB| = a$ be a side of a regular pentagon. Construct the regular pentagon $ABCDE$.

A regular pentagon consists of five identical isosceles triangles whose legs are the sides of the pentagon. That is why the problem can be change into construction an isosceles triangle with the legs a and the base $a(\frac{1+\sqrt{5}}{2})$. We also know the size of angles of the triangle: $36°$ at the base and $108°$ at the intersection of the legs.

Construction protocol

0. Points A, B; $|AB| = a$; circle $c; k(A, r = a)$
1. Circle $d; d(B, r = a)$
2. Point $C_1; C_1 \in c \cap d$
3. Point $D_1, I, K; loopsteps 1, 2, 3$
4. Circle $p; p(B, r = |BD_1| = a\sqrt{3})$
5. Circle $q; q(I, r = a\sqrt{3})$
6. Point $J; J \in p \cap q$
7. Circle $r; r(D_1, r = |AJ|)$
8. Circle $t; t(K, r = |AJ|$

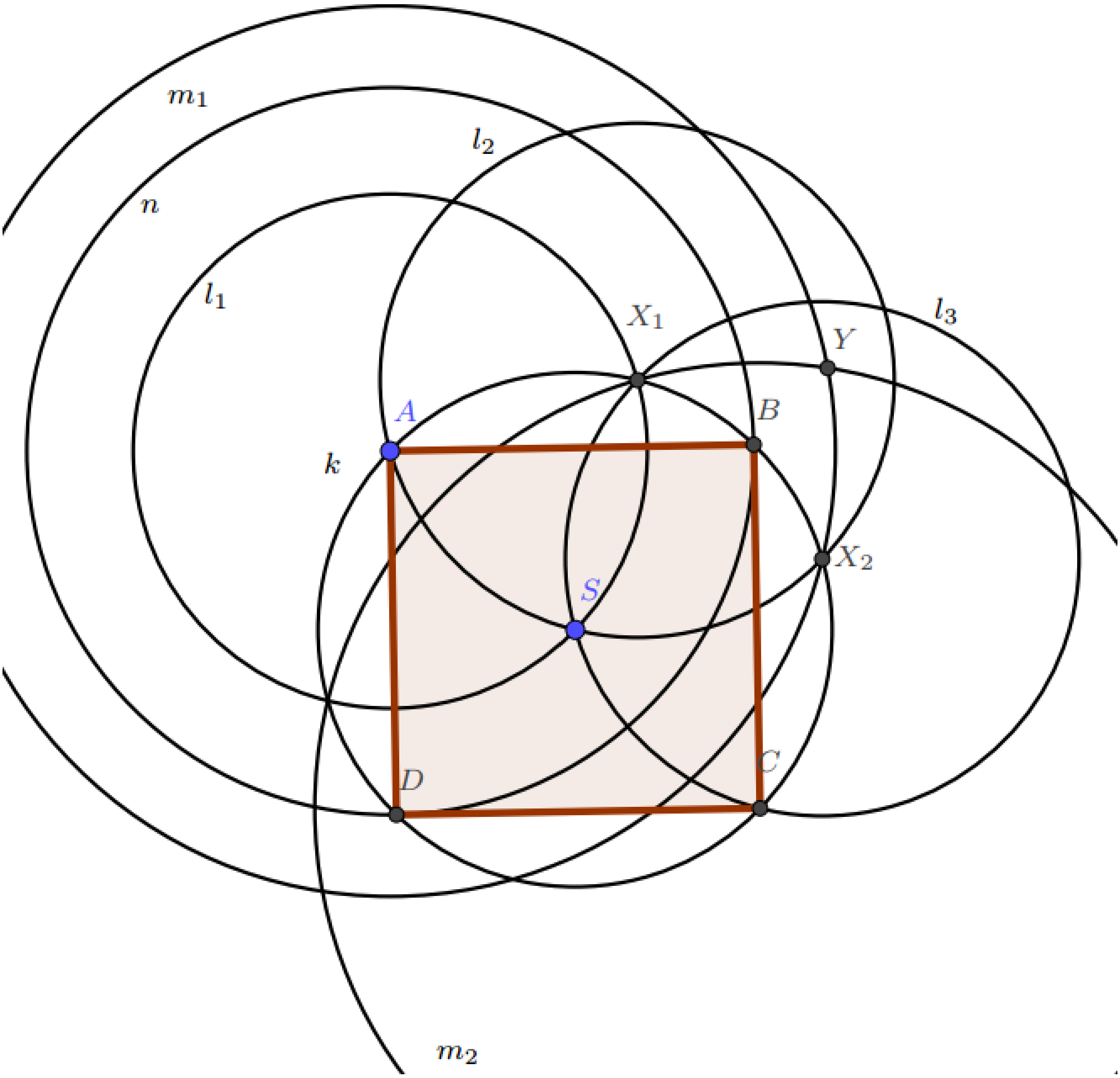

Fig. 4. The square $ABCD$ (A square of a diagonal u and an unknown side a)

9. Point $L; L \in r \cap t$
10. Circle $c_1; c_1(A; r = |IL|)$
11. Point $C; C \in d \cap c_1$
12. Pentagon $ABCDE$

During the construction, we use Mascheroni constructions of doubling a segment and division of an arch, Pythagorean theorem and the law of cosines. The aim is to construct a segment of length $a(\frac{1+\sqrt{5}}{2})$. We use the triangle BIJ in which $|BI| = |IJ| = |BD_1| = a\sqrt{3}$. Therefore the segment

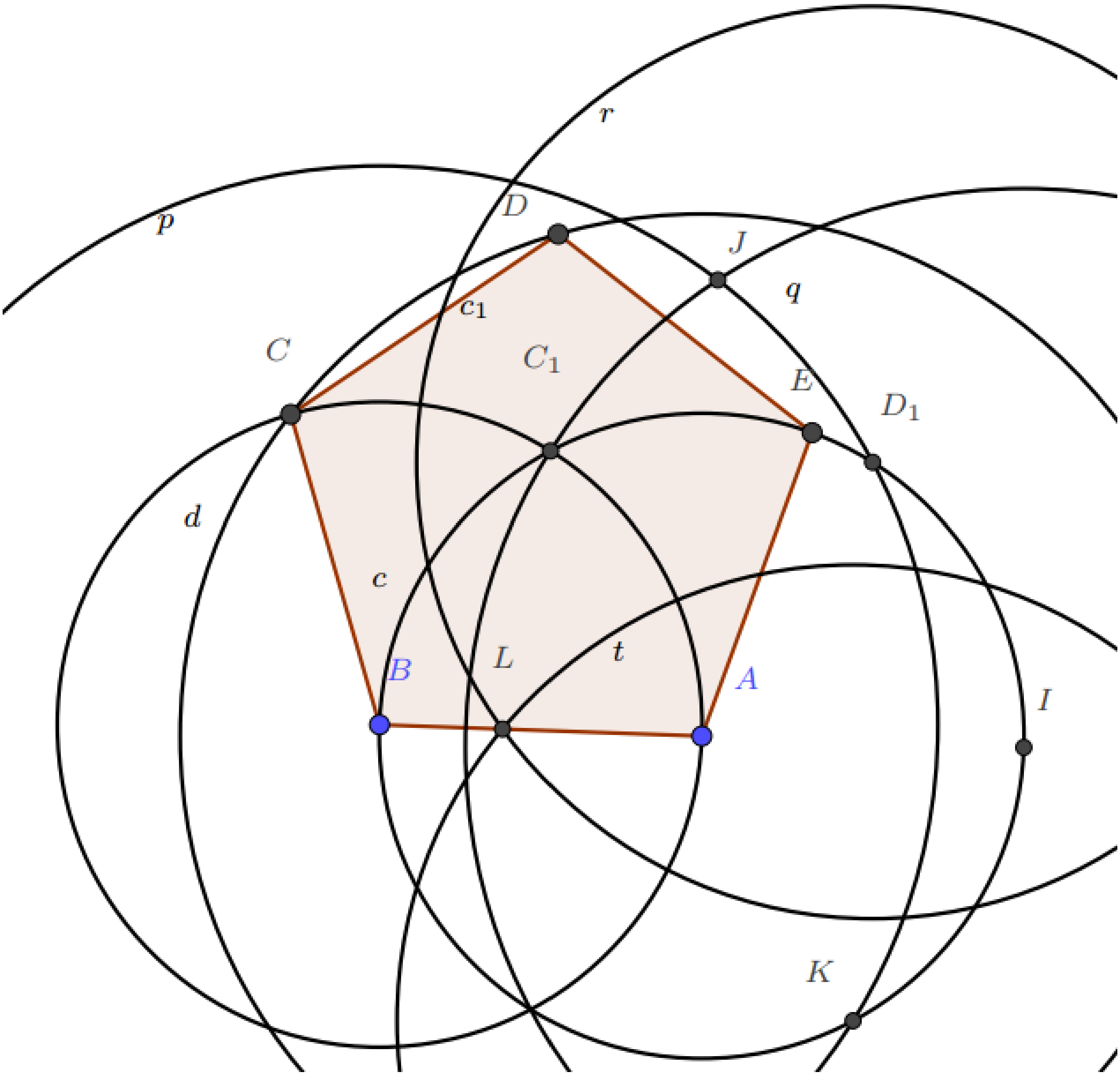

Fig. 5. The pentagon $ABCDE$

JA has the length $a\sqrt{2}$. Because $|D_1I| = |IK| = a$ a $|D_1K| = a\sqrt{3}$ for the triangle D_1IK, the length of IM[1] equals $\frac{1}{2}a$. As $|D_1L| = |KD_1| = \frac{\sqrt{3}}{2}a$ in the created triangle D_1LK, the length of LM equals $a\frac{\sqrt{5}}{2}$. Therefore $|IM|+|LM| = a(\frac{1+\sqrt{5}}{2})$ which is the segment we have been looking for.

[1] The point M is a center point of segment D_1K and is in the segment LI. We don't build the point M, because we use it only for a proof.

6 Conclusion

Mascheroni constructions were very popular in the past as seafarers or astronomers used them. Even today these constructions can appeal to us and can serve as motivational problems in constructional geometry. In the article, a few interesting construction of a square and a regular pentagon were introduced using GeoGebra. The constructions were slightly modified and may differ from the original ones in the book Geometria Del Compasso by Mascheroni.

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