Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

NEW TRENDS IN TEACHING FUNCTIONS AT UNIVERSITY LEVEL

DILLINGEROVÁ Monika (SK), VANKÚŠ Peter (SK)

Abstract. In our paper we focus on one of the key concepts of Mathematical Analysis – the functions. We present results of our survey, realized on the sample of 91 students. During this survey we analysed their solutions of selected tasks and identified some major misconceptions the students have in the area of the functions. In the paper we then propose some new trends those can help to avoid these misconceptions and improve students' knowledge.

Keywords: functions, mathematical analysis, misconceptions, teaching methods, ICT

Mathematics Subject Classification: Primary 97D40, 97C70; Secondary 97I99

1 Introduction

Authors of the paper are teaching Seminars on Mathematical Analysis at the Faculty of Mathematics, Physics and Informatics. In this paper we describe qualitative study of some mistakes the students makes in this topic and we propose some methods those can be helpful in overcoming these problems.

Functions are important concept that is essential for many other concepts in Analysis, so the good understanding of them is needed. Our paper surveys knowledge of our students in the area of functions and tries to point to the most common misconceptions they have. Afterwards, we propose various strategies and new trends of teaching. These should avoid described mistakes.

2 Theoretical framework

2.1 Analysis of students' misconceptions

Analysis of students' misconceptions is the process of identifying and reviewing students' errors to determine whether an error pattern exists – that means whether several students are making the same type of error. If a pattern does exist, the teacher can identify students' misconceptions or skill deficits and subsequently design and implement instruction to address them.

Research on error analysis is not new: Researchers around the world have been conducting studies on this topic for decades. Error analysis has been shown to be an effective method for identifying patterns of mathematical errors for any student, who is struggling in mathematics, as is stated in [1].

In the research process we have to analyze students' errors, attempt to understand them, explain what they consist in, and find what causes them. Depending on the conclusions of such an analysis, we should select corrective strategies and teaching methods in order to overcome these misconceptions.

Typically, student mathematical errors fall into three categories: factual, procedural, and conceptual. Each of these errors is related either to a student's lack of knowledge or a misunderstanding [2].

Factual errors occur when students lack factual information (e.g., definitions, formulas, vocabulary, digit identification, place value identification, etc.).
Procedural errors are associated with procedural knowledge and conceptual errors are associated with conceptual knowledge. Procedural knowledge refers to mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions. Conceptual knowledge refers to knowledge of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures [3], [4].

Not every error is the result of a lack of knowledge or skill. Sometimes, a student will make a mistake simply because he was fatigued or distracted (i.e., careless errors). Because conceptual and procedural knowledge often overlap, it is sometimes difficult to distinguish conceptual errors from procedural errors [5].

2.2 Analysis a priori and analysis a posteriori

Analysis a priori and analysis a posteriori are notions commonly used in the theory of didactic situations by Brousseau [6]. They are essential part of analysis of students' errors therefore we use them in our research.

Objective of analysis a priori of a mathematical task is to predict as accurately as possible the students' reactions and attitudes (obstacles, misconceptions and mistakes, correction of and further work with these mistakes), possible solving strategies (correct and incorrect), knowledge prerequisite for the use of the different solving strategies and also the teacher's reactions. E.g. it is the concept of an examiner work while making distractors to multiple-choice questions. In a posteriori analysis, a priori analysis is compared with experience from students' solutions of given task [7]. In our research we focus mostly on analysis of students' errors therefore this is the focus of both our analysis a priori and a posteriori of the tasks those were given to students.

3 Empirical results

3.1 Research design and analysis apriory

Our research was experiment with the sample of 91 students in the first semester of study at the Faculty of Mathematics, Physics and Informatics, Comenius University Bratislava. In the survey we did analysis of students' solutions of four tasks addressing the functions. Those tasks were part of a test from the subject Mathematical Analysis 1. All students wrote the test at the same time at 25.10. 2017, they all had the same variant of the test. We analyzed their results with the focus on the errors that students did and we summarized them based on the theoretical framework mentioned in previous chapter. This summarization was our further basis for recommendations of teaching strategies to overcome these errors and to strengthen students' knowledge of functions.

As the first stage of our qualitative analysis we did *analysis a priory* of the tasks.

Task 1: Composition of which elementary functions gives the function $y = \sqrt[3]{\dfrac{1}{\cos(x+2)}}$?

Use variables notions that indicate the order in which are functions composed.

Analysis apriory of the Task 1: We were expecting mostly good solutions; similar problems were part of seminars. Most common error we expected was incorrect order of functions during composition.

Task 2: State the function which graph we get from the graph of the function $y = \ln(2x-1)$ by:

a) Its shift by 3 units to the right.

b) Its shrink two-times (that means point $[x, y]$ we substitute with the point $\left[\dfrac{x}{2}, y\right]$).

We have standard position of the axes; we recommend testing your solutions for example by the positions of intersections with x axis.

Analysis apriory of the Task 2: This task is one of common tasks, which are part of mathematics already at upper secondary school. Therefore, we were expecting minimal mistakes mostly composed by the wrong direction of function shifting (left instead of right and stretching instead of shrinking).

Task 3: Sketch the graphs of these functions (each as one picture): $f: y = x^3$, $g: y = 0{,}3^x$, $F: y = \tan x$, $G: y = \log_{\frac{1}{2}} x$.

Analysis apriory of the Task 3: This task we perceived as the easiest one. Students just needed to use knowledge they already have from the upper secondary school and we used them also during our seminars. So we were expecting minimal errors.

Task 4: Sketch the graph of the function: $f(x) = \begin{cases} x-2, & x<0 \\ 2x+1, & x \geq 0 \end{cases}$

Find its range and inverse function f^{-1}.

Analysis apriory of the Task 4: The task is from our point of view not difficult. We were expecting minor problems with domain of the inverse function, but otherwise we believed students will have no problems to solve this task.

3.2 Analysis a posteriori

In this paragraph we present actual students' errors in selected tasks. We compare them with our analysis apriory and indicate main problems.

Task 1: 33 students solved the task correctly. 13 students did not solve the task at all. Most common errors of students are stated in table 1.

Name of error	Number of students	Type of the error
Some functions they used were not elementary	26	Factual error
They do not fully understand process of function composition	7	Conceptual error
They did not write the order in which functions are composed	5	Procedural error
They used wrong order of functions in composition	4	Conceptual error, Procedural error
They used bad way how to write the composition	3	Factual error

Tab. 1. Errors in the task 1.

Task 2: Part a) of the task was solved correctly by 16 students; 6 students did not solve it at all. Part b) was solved correctly by 20 students; 9 students did not even start to solve it. Most common errors of students are stated in table 2.

Name of error	Number of students	Type of the error
Part a): Students did remember they 'need to subtract 3 to move the graph to the right', but did that in the wrong way	51	Conceptual error
Part a): Students add 3 in different ways	9	Factual error, conceptual error
Part b): Students did remember they 'need to multiply by 2 to shrink the graph two-times', but did that in the wrong way	27	Conceptual error
Part b): Students divide by 2 in different ways	35	Factual error, conceptual error

Tab. 2. Errors in the task 2.

Task 3: 38 students graphed all 4 functions correctly. Table 3 gives an overview of the mistakes.

Name of error	Number of students	Type of the error
Bad shape of graph $y = 0{,}3^x$	26	Factual error
Bad shape of graph $y = \log_{\frac{1}{2}} x$	24	Factual error
Bad shape of graph $y = x^3$	19	Factual error
Bad shape of graph $y = \tan x$	5	Factual error

Tab. 3. Errors in the task 3.

Task 4: 24 students solved this task correctly. The mistakes of the students are listed in the table 4. In mistakes in graph of original function we commonly indentified that students did not change slope of linear function between $x - 2$ and $2x + 1$.

Name of error	Number of students	Type of the error
Bad domain of inverse function	36	Conceptual error
Bad range of original function	34	Conceptual error
Domain of the inverse function is missing	26	Conceptual error
Mistake in inverse function formula	22	Conceptual error, Procedural error
Mistake in graph of original function	17	Conceptual error, Procedural error

Tab. 4. Errors in the task 4.

3.3 Research results and discussion

In this chapter we will sum up results of our survey and discuss them.

Comparison between analysis a priori and analysis a posteriori in all tasks showed that students were doing much more errors that we have expected.

In tasks one the most common error was factual error that students did not know which functions are elementary ones. We also noticed conceptual error – some students did not understand process of function composition at all.

In task two the most common error was conceptual one. For example students remembered mechanically that they need ‘subtract 3 to move the graph to the right’ but they did that in wrong way: either just subtract three: $y = \ln(2x-1) \to y = \ln(2x-1-3)$; or subtracting 3 from the whole function: $y = \ln(2x-1) \to y = \ln(2x-1) - 3$.

Task 3 showed that students have factual problems even with most common graphs of elementary functions. This problem was the most visible when these functions were not the most common ones – in the case of logarithmic function and exponential function with the base lower than one.

In task 4 most students' errors were in finding domain of the inverse function, or not writing it altogether. The students have no conceptual connection that domain of function is necessary part of its definition. Students also showed some formal procedural errors in finding inverse function: some of them mechanically changed x with y but did not express inverse function formula. Many students also did mistake in the range of original function and its graph. In mistakes in graph of original function we commonly indentified conceptual error that students did not change slope of linear function between $x - 2$ and $2x + 1$.

These most common mistakes show us some deep conceptual, procedural and even factual errors in the area of functions. These errors can make obstacles in understanding other concepts in Mathematical Analysis. Therefore, in the next chapter of our article we will propose some trends in teaching to overcome these misconceptions and help students to get rid of them.

4 New trends of teaching to overcome students' errors

In this chapter we propose some strategies and problems those have potential to address issues found out during our survey. We will list them according the misconceptions they are connected with.

4.1 Shape of graphs

These strategies are connected with errors in changing the shape of graphs (task 2 and partially task 4 of our survey).

Students meet with drawing of functions graphs during their secondary education. But, the tasks they solve are more focused on making graphs from the function formulas. In our test we wanted the reverse operation. We knew that graph is shifted and shrank and we needed to find the function formula. In our survey we found out, that students did not have connection between shifting the graph by 3 units to the right and function formula $f(x-3)$.

To make this connection we propose to start with this introductory task:
1A) Sketch the graph of the function $f(x) = 5x$. Create the graph of the function $g(x) = f(x-3)$.

Important is also the connection to the formula of function $g(x)$, which means the ability to create formulas of new functions. Therefore, we need during the process of teaching to integrate tasks in which we have a formula and we have to sketch the graph. Nevertheless, we need also tasks when we have graph and formula of the function and we need to find formula of function with a changed shape. In this point it is important to use ICT technologies, because with just pen and paper we can manage only a few tasks. To make enough tasks in order to achieve deeper understanding of students we can sketch the graphs by computer, using e.g. freeware software Geogebra.

Thus, we can solve number of tasks with the problem:
1B) We have function $f(x)$. Sketch its graph and compare it with the graph of the function $g(x) = f(\text{some changes of argument})$.

Then, after solving several tasks of this type, we can move to the new tasks, again using ICT, in order to project the graphs to illustrate the solutions for the students. For example:
1C) In the figure 1 are graphs of $f(x)$ and $g(x)$. *a)* Write the formula of $g(x)$ using formula of $f(x)$, that means in format $g(x) = f(\text{some changes of argument})$. *b)* Using the points of its graph find the formula of $f(x)$. *c)* Using the points of its graph find the formula of $g(x)$. Compare your results with results from *a)* and *b)*.

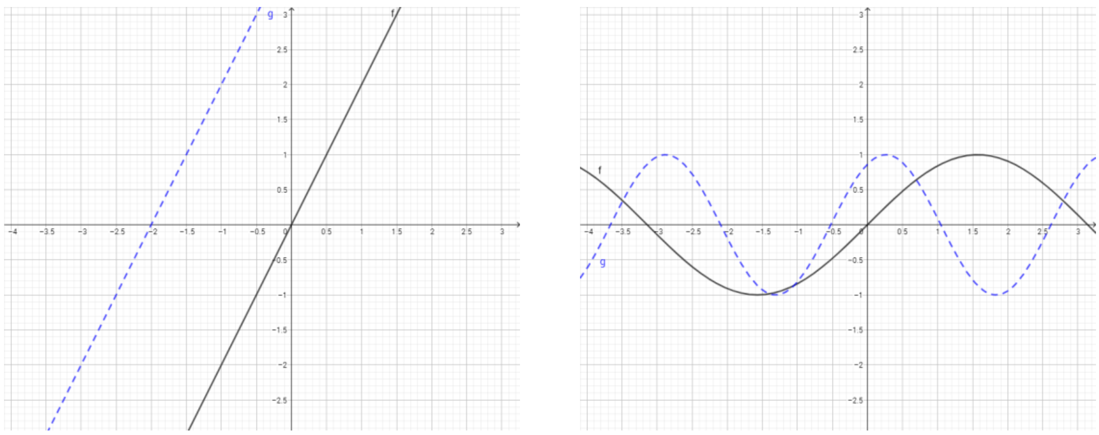

Fig. 1

During these tasks students find out importance of each aspect of function formula and graph. The need to be precise in these aspects, as e.g. the slope, is then apparent. This was a problem in task 4 of our survey.

4.2 Elementary functions

In this chapter we propose strategies to overcome errors connected with elementary functions and their graphs (task 1 and task 3 of our survey).

As we have seen in our survey, students do not understand notion elementary function. They meet this term first time at the university. It is not a part of any definition they need to know, it just begins to appear. Such usage of this notion without its clarification is considered propaedeutic in the theory of mathematics education. Our opinion is that this propaedeutic should come at secondary school and the creation of knowledge should come at the university level. This is factual error, so we need to fill this gap during our seminars. We need to point out that elementary function is the simplest form of function that we can call the class of the

functions. For example for the class of quadratic functions we have elementary function $f(x) = x^2$, which is the simplest quadratic function.
The tasks for students should look like:

2A) Which classes of functions do you know?

2B) What is the simplest form of square root function?

2C) Find out, which of the following functions are elementary:
$f(x) = \frac{1}{x}$, $g(x) = (x+1)^3$, $h(x) = \frac{1}{\cos x}$, $i(x) = \sin x$, $j(x) = \log_3 x$, $k(x) = \ln x$

Each of presented functions should be represented also by the sketch of its graph and verification of this graph using graphing software. This could be done during the presentation of the problems, if the teacher is skilled enough. Otherwise, the teacher should have prepared these graphs before the lesson in order to avoid unnecessary delays. The graphs could be then projected or created in the same time as we sketch them on the blackboard.

In order to get rid of the errors in the task 3, connected with the graphs of exponential and logarithmic function with argument lower than 1, the students should meet with these functions more often. Here can help just careful selection of easy tasks leading to sketching of such graphs during the seminars.

Another problem for students was the composition of functions. We have partially discussed this already in previous chapter. Now is the time to focus students' attention on this. We can add in the task 2C instruction: If some of the functions is not elementary one, write the elementary functions whose composition creates this function. E.g. for function $g(x) = (x+1)^3$ we should get functions $y_1 = x^3$, $y_2 = (x+1)$. How to composite them can be again find out with the help of computer. We will create graphs of $g(x), y_1, y_2, y_1 \circ y_2, y_2 \circ y_1$ and we will compare compositions of functions with the graph of $g(x)$. The teacher should have all these graphs prepared, see figure 2.

Fig. 2

In the same way we solve also function $h(x)$ from the task 2C. Then, we continue with more complex functions. The process of the composition of the functions should be always checked by the graphs! If we compose three functions, we should have prepared graphs of possible compositions and actual correct function.

For example task:

2D) Find out elementary functions those give as composition the function $y = \dfrac{1}{\sin(x - \frac{\pi}{4})}$ and write, how you need to compose them.

In this task we should get the functions $y_1 = x - \frac{\pi}{4}$, $y_2 = \sin x$, $y_3 = \frac{1}{x}$. Students will probably state as correct ones compositions $y_1 \circ y_2 \circ y_3$ and $y_3 \circ y_2 \circ y_1$. Teacher needs the graphs of these two compositions and the graph of the given function. This verification of the result is for students very important. Such graphical representations will help avoid students' misconceptions discussed in this chapter.

4.3 Domain of function and inverse function

In this chapter we will deal with errors in the task 4 of our survey. These are connected with the domain of the functions and also formal problems in the creation of inverse function.

Domain of the function appears to be not useful notion in the beginning of the secondary school mathematics education. Students meet at first with linear and quadratic functions. Importance of the domain is apparent just with logarithmic and goniometric functions. Controversially, in the whole document ISCED 3A (Educational program for upper secondary school education) [8] we cannot find the notion domain of the function. In the textbook for second year of upper secondary education [9] in the chapter about the graph of quadratic functions (page 50) is nothing about the domain or the range of these functions. In this way domain does not become automatically part of the function for the students.

However, despite these facts, the domain of the function and its range is part of upper secondary mathematics education. These concepts are included in the external tests from mathematics during the upper secondary school leaving exams. For example these tasks were directly focused on the domain of the function: year 2016, test 5178, task 24; year 2015, test 1203, task 21; year 2013, test 8103, task 27; year 2011, test 3306, task 25; etc. [10]

As explained, the domain and range of the function are not perceived by the students as integral part of the function. We should accept this situation at our seminar and focused the tasks on them – read them from the graphs of functions and also look for domain of each function we deal with from its formula.

Formal problems in the creation of the inverse function can be avoided by the application tasks from physics which do not use variables *x* and *y*.

Typically, these tasks can look like:
3A) Write the formula for the distance (in meters) depending on the time (in seconds) when object performs uniform motion at the speed 5 ms^{-1}. Sketch the graph of this relationship. Then, write the relationship of the time depending on the distance in the same motion and graph it. What did you noticed comparing these two graphs?

Similarly, we can do the same with the formula of the free fall motion, when the function is not just a linear one ($h = \frac{1}{2}gt^2$). We can find a lot of such problems in the physics. In this way students will see necessity to express variable from the formula and will be not fixed just on variables x and y. The task for the teacher is to point out that these problems are connected with the inverse functions.

5 Conclusion

In our paper we presented our survey, realized on the sample of 91 students in the first semester of university study at Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava. This survey was focused on functions as one of very important concepts in the university level mathematics. We analysed students' solutions of selected mathematical tasks and identified some major misconceptions connected with the functions. In order to avoid these misconceptions we then presented some trends of the teaching. These trends include proposals of the problems, instructions for using of ICT in the education of this topic and also recommendations for teachers useful to improve students' knowledge.

References

[1] BROWN J., SKOW K., IRIS CENTER, *Mathematics: Identifying and addressing student errors.* 2016, URL, http://iris.peabody.vanderbilt.edu/case_studies/ics_matherr.pdf.

[2] FISHER, D., FREY, N., Making time for feedback. *Feedback for Learning*, 70(1), p. 42–46.

[3] EISENHART, M., BORKO, H., UNDERHILL., R., BROWN, C., JONES, D., AGARD, P., Conceptual knowledge fall through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24(1), 1993, p. 8–40, ISSN 0021-8251.

[4] MORU, E.K., QHOBELA, M., POKA, W., NCHEJANE, J., Teacher knowledge of error analysis in differential calculus. *Pythagoras*, 35(2), 2014, ISSN 2223-7895.

[5] RICCOMINI, P. J., *Identifying and using error patterns to inform instruction for students struggling in mathematics*. Webinar series, 2014, URL, http://www.ohioregion14.org/perspectives/?p=1005

[6] BROUSSEAU, G., *Theory of Didactical situations in mathematics 1970-1990*. [Edited and translated M. Cooper, N. Balacheff, R. Sutherland and V. Warfield.], Dordrecht: Kluwer Academic Publishers, 1997.

[7] MORAOVÁ, HANA; NOVOTNÁ, J., Lesson planning in CLIL in the Light of Brousseau's Theory of Didactical Situations in Mathematics. *Sborník konference Kompetence v cizích jazycích jako důležitá součást profilu absolventa vysoké školy IV.,* Brno: Univerzita obrany, 2014, p. 203 - 210.
[8] URL, http://www.statpedu.sk/files/articles/dokumenty/statny-vzdelavaci-program/matematika_isced3a.pdf.
[9] KUBÁČEK, Z., *Matematika pre druhý ročník gymnázií. Prvá časť*. Bratislava: Orbis Pictus Istropolitana, 2009.
[10] URL, https://www.matura.sk/maturitne-testy/matematika.

Current address

Dillingerová Monika, RNDr., PhD.
Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava
Mlynská dolina, 842 48 Bratislava, Slovak Republic
E-mail: monika.dillingerova@fmph.uniba.sk

Vankúš Peter, PaedDr., PhD.
Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava
Mlynská dolina, 842 48 Bratislava, Slovak Republic
E-mail: peter.vankus@fmph.uniba.sk