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DO IMPORTANT NETWORK ACTORS FORM IMPORTANT TIES?

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Abstract. In this contribution, we analyze the location of vertices and edges with respect to maximum betweenness centrality within a graph. We present a construction of graphs where the distance of the set of vertices and the set of edges with the maximum betweenness (the betweenness separation) can be arbitrarily high, together with sufficient conditions for graphs in which this distance is zero. We also discuss the betweenness separation in real-world networks and several models of pseudo-random graphs which use to model complex networks.

Keywords: betweenness, edge betweenness, prism of graph, random graph, Barabási-Albert model

Mathematics subject classification: Primary 05C82; Secondary 05C12, 91D30

1 Introduction

Throughout this paper, we consider connected simple graphs (without loops or multiple edges). In a graph $G = (V, E)$, the distance $d_G(u, v)$ of two vertices $u, v \in V$ is the length of the shortest $u - v$-path in G. For two sets $X, Y \subseteq V$, we define $d_G(X, Y) = \min_{u \in X, v \in Y} d_G(u, v)$. The diameter $\mathrm{diam}(G)$ of G is the largest distance of vertices of G. Other notation and terminology used here is taken from [12].

When analyzing a social network from the point of view of information spreading, one is interested in determining both the actors through which the most of information flows (because then they get more informed and have more possibilities to control this flow) as well as the links most frequently used in mutual actor communication (because its breaking can badly influence the access of actors to information). As suitable measure of amount of information flow through actors and links, one may consider betweenness centrality indices, which are defined in the following way (see [3] and [7]): for a vertex x and an edge yz of a graph $G = (V, E)$, the vertex betweenness of x is

$$B_G(x) = \sum_{\substack{u,v \in V \\ u \neq x \neq v}} \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}}$$

where $\sigma_{u,v}^G$ is the number of shortest paths of G between two distinct vertices u, v and $\sigma_{u,v}^G(x)$ is the number of those shortest $u-v$-paths in G having x as an internal vertex (the subscript is omitted if G is clear from context). The edge betweenness of yz is

$$B_G(yz) = \sum_{u,v \in V} \frac{\sigma_{u,v}^G(yz)}{\sigma_{u,v}^G}$$

where $\sigma_{u,v}^G(yz)$ is the number of those shortest $u-v$-paths of G that pass through yz (here u and v may coincide with y, z). These invariants are often applied in social network analysis and, also, their graph-theoretical properties are recently investigated, see [4], [5], [10] and the survey chapter [6].

Within a network, an actor with the highest betweenness appears on many shortest paths, and thus one might expect that, of the links that this actor forms, some shall have high edge betweenness too (or, less formally, important actors should form also important links). Formally, we quantify the "closeness" of vertices and edges with the maximum vertex / edge betweenness in a graph G by *betweenness separation* $\mathfrak{F}(G)$ which is equal to the distance $d_G(\mathcal{B}_v(G), \mathcal{B}_e(G))$ where $\mathcal{B}_v(G)$ is the set of all vertices of G with the maximum vertex betweenness, and $\mathcal{B}_e(G)$ is the set of endvertices of all edges of G with the maximum edge betweenness. While we show that, in general, the value of $\mathfrak{F}(G)$ may be arbitrarily large, we present the results of computations on large collections of graphs randomly generated according to particular graph distributions which suggest that, for real-world networks, the above paradigm is often true or, at least, the vertices and edges of maximum betweenness are not too distant.

2 Theoretical results

In this section, we present several results on the values of $\mathfrak{F}$ for graphs in general.

Example 1: Consider the graph K_n^{--} ($n \geq 5$) obtained from the complete graph K_n on the vertex set $\{v_1, \ldots, v_n\}$ by deleting its two nonadjacent edges, say v_1v_2 and v_3v_4. Observe that only two pairs of vertices – namely, $\{v_1, v_2\}$ and $\{v_3, v_4\}$ – add a positive contribution (equal to $\frac{1}{n-2}$) to the betweenness of a vertex of K_n^{--}; thus, the betweenness of vertices of K_n^{--} is $\frac{2}{n-2}$ or $\frac{1}{n-2}$, and there are $n-4$ vertices of maximum betweenness (the ones with the maximum degree). Similarly, for an edge v_iv_j of K_n^{--}, the only positive contributions to its edge betweenness come from the pair $\{v_i, v_j\}$ (the value 1) and, possibly, from the pairs $\{v_1, v_2\}$ and $\{v_3, v_4\}$ (each with the value $\frac{1}{n-2}$). We then conclude that the maximum betweenness of edges of K_n^{--} is $1 + \frac{1}{n-2} + \frac{1}{n-2} = \frac{n}{n-2}$ and is attained for the four edges with endvertices from $\{v_1, v_2, v_3, v_4\}$. This example also shows that the betweenness separation can be nonzero even in very dense graphs. ■

Next, we present the construction of graphs which have arbitrarily large betweenness separation.

Example 2: Given positive integers $n, d \geq 2$, let $B_{n,d}$ be the graph obtained from the n-vertex path $x_1x_2 \ldots x_n$ by connecting the vertex x_n to d new vertices $y_1, \ldots, y_d$. By routine calculations, we compute the betweenness of vertices and edges of $B_{n,d}$:

$B(x_1) = B(y_i) = 0$ for each $i = 1, \ldots, d$,
$B(x_i) = (i-1)(n+d-i)$ for each $i = 2, \ldots, n-1$,
$B(x_n) = \binom{d}{2} + d(n-1)$,
$B(x_ny_i) = n+d-1$ for each $i = 1, \ldots, d$,
$B(x_ix_{i+1}) = i(n+d-i)$ for each $i = 1, \ldots, n-1$

Let k be a positive integer. Choose now n, d such that $n - d > 2(k + 1)$ and $d > \frac{n}{2}$. Then the maximum edge betweenness is attained for the edge $x_i x_{i+1}$ with $i = \frac{n+d}{2}$ if $n + d$ is even, or for the pair of edges $x_{i-1}x_i, x_i x_{i+1}$ with $i = \frac{n+d-1}{2}$ otherwise. Furthermore, for $i = 2, \ldots, n-1$, the maximum of $B(x_i)$ is equal to $\left(\frac{n+d-1}{2}\right)^2$ being attained for $i = \frac{n+d+1}{2}$; the condition $d > \frac{n}{2}$ now implies that, for $i = 2, \ldots, n-1$, $B(x_n) - \max_{i=2,\ldots,n-1} B(x_i) \geq \binom{d}{2} + d(n-1) - \left(\frac{d+n+1}{2}\right)^2 = \frac{d^2}{4} + \frac{dn}{2} - d - \frac{n^2}{4} + \frac{n}{2} - \frac{1}{4} > \frac{n^2}{16} - \frac{1}{4} \geq 0$ since $n \geq 2$. Thus x_n has the maximum vertex betweenness among all vertices of $B_{n,d}$. The distance of the vertex x_n and the maximum betweenness edge is equal to $n - \left(\frac{n+d}{2} + 1\right) = \frac{n-d-2}{2} > k$. ■

On the other hand, the betweenness separation of many graphs is equal to zero. This is clearly true for vertex-transitive or edge-transitive graphs, and for nontransitive graphs of many specific families. For example, examining (by help of Wolfram Mathematica) the grids (that is, the Cartesian products $P_m \Box P_n$ of two paths on m and n vertices), one finds that the set $\mathcal{B}_v(P_m \Box P_n)$ consists either of the single vertex, or pair of adjacent vertices, or else four vertices forming 4-cycle (depending on whether m, n are both odd, or exactly one of them is odd, or both are even), which corresponds to Cartesian product of graphs induced by $\mathcal{B}_v(P_m)$ and $\mathcal{B}_v(P_n)$ (the central vertex or the pair of central vertices in path). Similarly, one finds that $\mathcal{B}_e(P_m \Box P_n)$ consists either of four edges with common endvertex (when $m = n$ is odd), or four edges forming 4-cycle (when $m = n$ is even), or two nonadjacent edges (when $m \neq n$, both being even), or two adjacent edges (when $m \neq n$, both being odd), or else a single edge. In each case, one obtains $\mathfrak{G}(P_m \Box P_n) = 0$. An analogous behaviour may be observed also in higher–dimensional grids.

The above observations for grids suggest the following sufficient condition for zero betweenness separation:

Lemma 1 *Let G be a connected k-regular graph such that the subgraph induced on $\mathcal{B}_e(G)$ contains k edges with a common vertex. Then $\mathfrak{G}(G) = 0$.*

Proof: We use the fact (see [6]) that, for any vertex $x \in V(G)$,

$$B(x) = \frac{1}{2}\left(\sum_{xy \in E(G)} B(xy) + 1 - |V(G)|\right).$$

From this, it follows that $\max_{x \in V(G)} \sum_{xy \in E(G)} B(xy)$ is attained for a vertex z of $\mathcal{B}_e(G)$ having k neighbours in $\mathcal{B}_e(G)$; thus z has the maximum betweenness among vertices of G and is incident with edges of maximum edge betweenness. □

In this lemma, the number of k maximum betweenness edge with a common vertex cannot be decreased – the cubic graph G on Figure 1 has $\mathcal{B}_v(G) = \{8, 10\}$ and $\mathcal{B}_e(G) = \{\{3, 7\}, \{7, 12\}\}$.

The comparison of sets of graph elements with maximum betweenness in graphs and their Cartesian products might also suggest that, for any connected graphs G and H, $\mathcal{B}_v(G \Box H) = \mathcal{B}_v(G) \times \mathcal{B}_v(H)$ and $\mathcal{B}_e(G \Box H) \subseteq \mathcal{B}_e(G) \times \mathcal{B}_e(H)$ (the symbol $\times$ here refers to Cartesian product of sets). However, these relations are not true in general: in the graph on Figure 2, $\mathcal{B}_v(G) = \{1\}$ and $\mathcal{B}_e(G) = \{\{1, 5\}\}$ while $\mathcal{B}_v(G \Box K_2) = \{2, 9\}$ and $\mathcal{B}_e(G \Box K_2) = \{\{2, 9\}\}$ (the vertex 9 being the counterpart of the

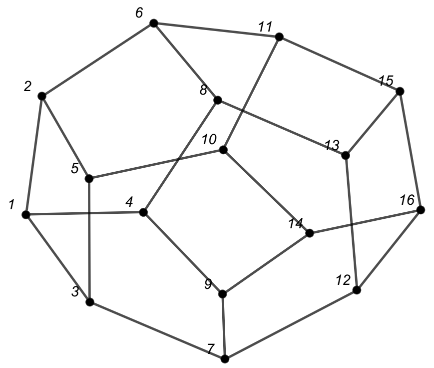

Fig. 1. A cubic graph G with $\mathfrak{d}\!\mathfrak{z}(G) = 0$ with $\mathcal{B}_e(G)$ consisting of two adjacent edges

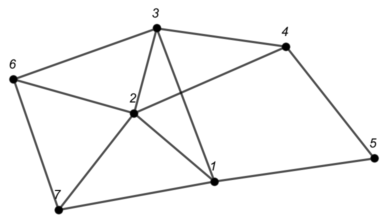

Fig. 2. A graph G whose $\mathcal{B}_v(G)$ and $\mathcal{B}_e(G)$ does not extend to $\mathcal{B}_v(G\square K_2)$ resp. $\mathcal{B}_e(G\square K_2)$

vertex 2 in the second copy of G in $G\square K_2$). Using the Wolfram Mathematica computer algebra system, we have verified that this is the unique smallest graph with such property.

These relations are generally not true even for trees, as seen from the example on Figure 3: here $\mathcal{B}_v(T) = \{6\}$ while $\mathcal{B}_v(T\square K_2) = \{1, 16\}$ (the vertex 16 being the counterpart of the vertex 2 in the second copy of T in $T\square K_2$). We have also verified that this is the smallest tree with such property, besides other three trees on the same number of vertices.

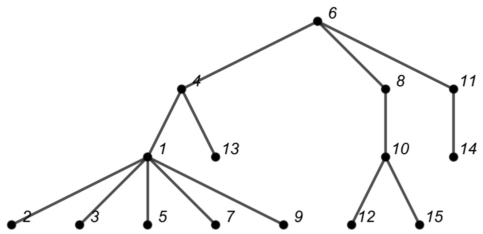

Fig. 3. A tree T whose $\mathcal{B}_v(T)$ does not extend to $\mathcal{B}_v(T\square K_2)$

More precise information on values of vertex betweenness in graph prisms is provided by

Theorem 2 *Let G be a connected graph and $x \in V(G\square K_2)$. Then*

$$B_{G\square K_2}(x) = 2B_G(x) + \sum_{u,v \in V(G)\setminus\{x\}} \left(\frac{1}{d_G(u,v)+1}\right) \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}} + \sum_{u \in V(G)\setminus\{x\}} \frac{1}{d_G(u,x)+1}.$$

Proof: Set $H = G\square K_2$ with $V(H) = V(G_1) \cup V(G_2)$ where G_1, G_2 are layered copies of G in H. Without loss of generality, let $x = x_1 \in V(G_1)$, $x_2 \in V(G_2)$ be the counterpart of x in G_2 (in general, to distinguish the vertices of G_1 and G_2 which are counterparts of each other, we will use subscripts 1 and 2). Then

$$B_H(x) = \sum_{u,v \in V(H)\setminus\{x\}} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} = \sum_{u,v \in V(G_1)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} + \sum_{u,v \in V(G_2)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} + \sum_{u \in V(G_1), v \in V(G_2)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} =$$

$$A + B + C.$$

If $u, v \in V(G_1)$ then all shortest $u - v$-paths in H are contained in G_1, thus

$$A = \sum_{u,v \in V(G_1)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} = \sum_{u,v \in V(G_1)} \frac{\sigma^{G_1}_{u,v}(x)}{\sigma^{G_1}_{u,v}} = B_G(x)$$

and (because $x \in V(G_1)$)

$$B = \sum_{u,v \in V(G_2)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} = 0.$$

Now write

$$C = \sum_{u \in V(G_1), v \in V(G_2)} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} = \sum_{u \in V(G_1)\setminus\{x\}, v \in V(G_2)\setminus\{x\}} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} + \sum_{u \in V(G_1)\setminus\{x\}} \frac{\sigma^H_{u,x_2}(x)}{\sigma^H_{u,x_2}}$$

If $\sigma^G_{u,v}(x) = 0$ for some $u, v \in V(G)$, then $\sigma^H_{u_1,v_2}(x) = 0$. Let $\sigma^G_{u,v}(x) > 0$ and $d_G(u,v) = d = i + j$ where $i = d_G(u,x)$ and $j = d_G(x,v)$. Then $\sigma^H_{u_1,v_2} = (d+1)\sigma^G_{u,v}$ (because any shortest $u - v$-path of G may be considered as a "projection" of certain $d+1$ shortest $u_1 - v_2$-paths of H) and, similarly, $\sigma^H_{u_2,v_1} = (d+1)\sigma^G_{u,v}$, $\sigma^H_{u_1,v_2}(x) = (j+1)\sigma^G_{u,v}(x)$, $\sigma^H_{u_2,v_1}(x) = \sigma^G_{u,v}(x)$. Hence

$$\sum_{u \in V(G_1)\setminus\{x\}, v \in V(G_2)\setminus\{x\}} \frac{\sigma^H_{u,v}(x)}{\sigma^H_{u,v}} = \sum_{u,v \in V(G)\setminus\{x\}} \left(\frac{j+1}{d+1} \cdot \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}} + \frac{i+1}{d+1} \cdot \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}} \right) =$$

$$\sum_{u,v \in V(G)\setminus\{x\}} \frac{d+2}{d+1} \cdot \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}} = B_G(x) + \sum_{u,v \in V(G)\setminus\{x\}} \frac{1}{d+1} \cdot \frac{\sigma^G_{u,v}(x)}{\sigma^G_{u,v}}$$

Moreover, for each $u_1 \in V(G_1)$, exactly one shortest $u_1 - x_2$-path passes through x, so

$$\sum_{u \in V(G_1)\setminus\{x\}} \frac{\sigma^H_{u,x_2}(x)}{\sigma^H_{u,x_2}} = \sum_{u \in V(G_1)\setminus\{x\}} \frac{1}{d(u,x)+1}.$$

Summing the obtained expression for A, B, C then yields the result. □

Note that, in the above expression for $B_{G\Box K_2}(x)$, the middle and the last sum are very similar to length-scaled betweenness centrality (introduced in [2], see also [6]) and the reciprocal centrality (see [9]) – indeed, their denominators differ just by the additive factor 1. Thus, the mutual relation of $\mathcal{B}_v(G\Box K_2)$ and $\mathcal{B}_v(G)$ is influenced by the mutual location of the central vertices with respect to betweenness, length-scaled betweennes and the reciprocal centrality (where one can expect that there exist graphs in which the sets of those central vertices are mutually disjoint).

Network	$\mathfrak{z}$	diameter	order	size
ZacharyKarateClub	0	5	34	78
DolphinSocialNetwork	0	8	62	159
DavisSouthernWomen	1	4	32	89
EurovisionVotes	1	∞	46	467
JazzMusicians	1	6	198	2742
LesMiserables	0	14	77	254
September11Terrorists	0	5	37	85
TaggedTestImages	3	8	71	188
USPoliticsBooks	2	7	105	441
WordAdjacencies	0	5	112	425
WorldCup1988	0	∞	35	118
AmericanCollegeFootball	2	4	115	613

Tab. 1. Betweenness separation of selected real-world networks (the individual networks are obtained, in Wolfram Mathematica, by evaluating the command `ExampleData[{"NetworkGraph","<network_name>"}]` where `<network_name>` is an entry from the first column)

3 Experimental results

It is interesting that, for many real-world networks, the betweenness separation is zero or very small compared to network characteristics. This claim is supported by exhibiting the selected graphs from the collection of example graphs provided by Wolfram Research data server (their full list can be obtained, in environments supporting Wolfram Language (like Wolfram Mathematica or Wolfram Programming Lab), by evaluating the procedure `ExampleData ["NetworkGraph"]`; in total, there are 228 networks, and we studied those ones having not many vertices). The obtained results are summarized in Table 1.

In addition, we tested large collections of graphs generated for various graph distributions, some of them being related to real-world networks. Particularly, three graph distributions were considered:

- Uniform graph distribution with parameters n, m – corresponds to the set of all graphs on n vertices and m edges.
- Bernoulli graph distribution with parameters n, p – the graphs are constructed from n independent vertices by selecting each edge independently with probability p.
- Barabási-Albert graph distribution with parameters n, k – the graphs are constructed by consecutive adding new vertices of degree k (as an initial graph, one can take, for example, a $(k + 1)$-cycle) until reaching n vertices in total in such a way that the k edges of the newly added vertex are attached to existing vertices at random, but with the probability proportional to the number of edges that the existing nodes already have.

For each of these graph distributions, we generated, using pseudo-random generating procedures provided by Wolfram Mathematica, lists of 10000 graphs on 30 and 60 (and, for the Barabási-Albert distribution, on 120) vertices. The numbers of vertices were chosen to reach a compromise between the performance of betweenness separation testing algorithm and the need to work with large number of not-so-trivial graphs (while sparse graphs were relatively easy to handle, for random graphs with high edge density, the algorithm has performed slowly). The lists were generated in straightforward way by evaluating the commands `RandomGraph[{m,n},10000]` for pseudo-random graphs on n vertices and m edges, and

```
Table[RandomGraph[BernoulliGraphDistribution[n, p]],10000]
```

or

```
Table[RandomGraph[BarabasiAlbertGraphDistribution[n,k]],10000]
```

for random n-vertex graphs with edge probability p or n-vertex graphs resulted from Barabási-Albert process, respectively. For uniform graph distribution, we have also chosen different values of m (50, 100, 200, 300, 400 and $\binom{30}{2} - 2$ for 30-vertex graphs, and twice much edges for 60-vertex graphs) and, for Bernoulli graph distribution, four edge probabilities p ($0.1, 0.25, 0.5$ and 0.75). The Barabási-Albert graph distribution was considered with parameter $k = 2, 3, 5, 10$.

The histograms for distribution of $\mathfrak{z}(G)$ in uniform distribution-generated graphs are shown on Figure 4. One may observe that $\mathfrak{z}$ is small in general, and tends to be mostly 0 for sparse graphs. But, when a graph gets more edges, its betweenness separation increases, reaching eventually a peak for certain edge density after which its values decrease again; if a graph G is just two edges apart from the complete one, then $\mathfrak{z}(G) \in \{0, 1\}$ with majority of graphs having the value 1. This is consistent with the fact that the fraction of labelled n-vertex graphs with $\binom{n}{2} - 2$ edges is equal to $\frac{\frac{1}{2}\binom{n}{2}\binom{n-2}{2}}{\binom{\binom{n}{2}}{2}} \to 1$ for $n \to +\infty$.

Figure 5 contains the results for randomly generated graphs with different edge probability. The simulations suggest that the edge probability at least $\frac{1}{2}$ yields the betweenness separation equal to 1 for majority of random graphs regardless on the number of vertices; for low probabilities, however, the frequency of individual values for $\mathfrak{z}(G)$ seems to depend also on the order of G. Nevertheless, the overall values again tend to be small.

For pseudo-random graphs generated from Barabási-Albert graph distribution (see Figure 6), one can also observe the trend for small values of $\mathfrak{z}$, with majority of graphs having zero value; nevertheless, with the increasing degree of vertices being added, significant portion of graphs posesses nonzero values.

4 Concluding remarks

The above results show that to answer the question in this paper's title is not easy if we choose the betweenness as the measure of importance of actors and ties. The situation perhaps might be less complicated under another centrality measure, however, one has to consider the vertex centrality which also extends, in some generic way, to the edges. So far, we are aware only of one other such centrality, namely the current-flow betweenness, see [1]. It would be interesting to consider an analogue of vertex closeness $C(x)$ (defined as the reciprocal of the sum of distances from selected vertex x to every other vertex) for edges (could be defined, for example, as the sum of closenesses of edge endvertices, or as the sum of all distances from the set of ednvertices of given edge to sets of endvertices of all other edges) and to define, in similar way as at the beginning of Section 2, the closeness separation between the graph elements of the maximum closeness in connected graphs. Note that, in contrast with Theorem 2, it is easy to show that $C_{G\square K_2}(x) = \frac{1}{2\frac{1}{C_G(x)} + |V(G)|}$ and

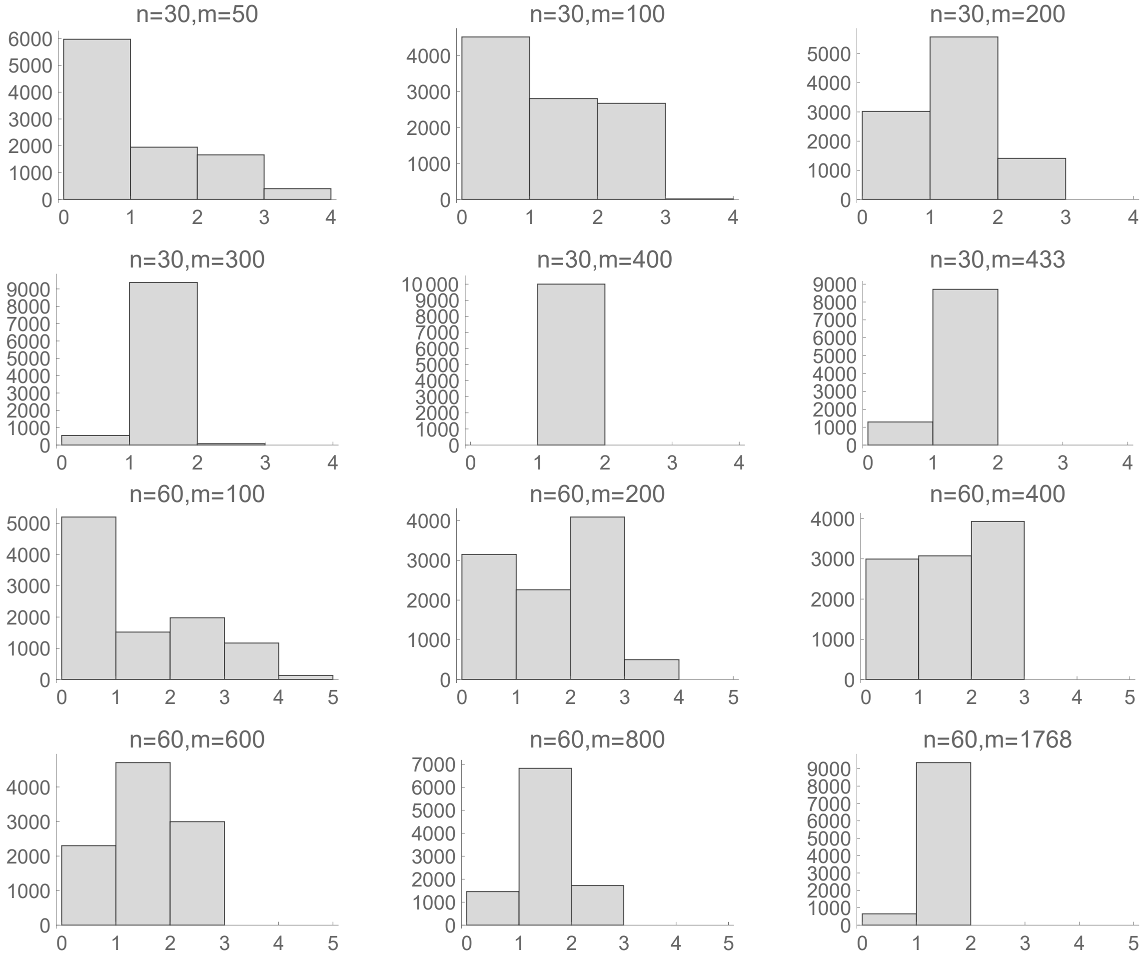

Fig. 4. Histograms for uniformly generated graphs

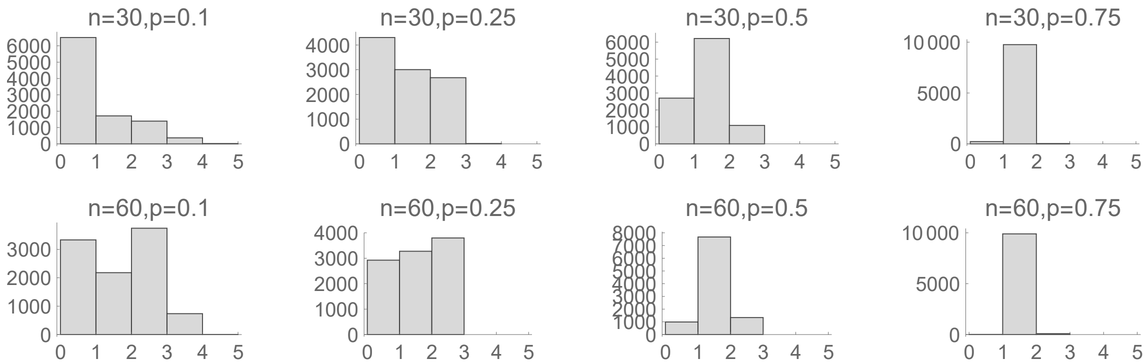

Fig. 5. Histograms for random graphs

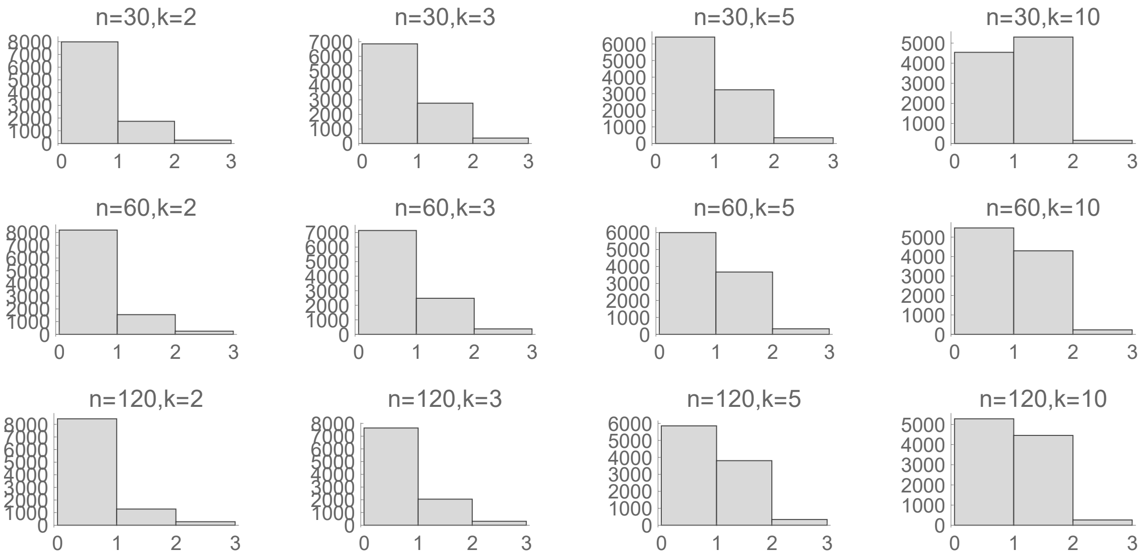

Fig. 6. Histograms for graphs of Barabási-Albert distribution

thus, the set of vertices of highest closeness in G naturally lifts into analogous set in the prism of G; therefore, one might hope that the use of closeness centrality instead of betweenness would result in problems being easier to handle.

As the betweenness centrality is often used in analyses of biological networks like protein-protein interaction networks (see for example [8]), it would be interesting to examine the betweenness separation of them. The relative sparsity of some of these networks would allow the reasonable use of our developed Wolfram Mathematica framework; note, however, that for more complicated examples, one should probably use, instead of built-in betweenness algorithms, a more efficient code.

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