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CONFORMAL MAPPINGS OF RIEMANNIAN MANIFOLDS PRESERVING THE GENERALIZED EINSTEIN TENSOR

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Abstract. We study conformal mappings preserving the generalized Einstein tensor. We have derived corresponding partial differential equations and their integrability conditions. In addition to the generalized Einstein tensor we got other invariants of the mappings. Also we have proved that orientable compact manifolds equipped by positive definite metric, do not admit conformal mappings preserving the generalized Einstein tensor.

Keywords: Affine connection, Riemannian manifolds, conformal mappings, generalized Einstein tensor

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1 Introduction

Diffeomorphisms preserving certain geometric objects are being given much attention of many researchers in the differential geometry realm.

In particular, conformal mappings which preserved the Einstein tensor

$$E_{ij} = R_{ij} - \frac{Rg_{ij}}{n}$$

studied in [1]. Preserving the stress-energy tensor

$$S_{ij} = R_{ij} - \frac{Rg_{ij}}{2}$$

by conformal mappings was explored in [4], [2]. It's worth for noting that in many classical issues e. g. [7, p. 359], just the latter is referred to as the Einstein tensor. Let us refer to

$$\mathfrak{E}_{ij} \stackrel{\text{def}}{=} R_{ij} - \kappa R g_{ij}. \tag{1}$$

as **the generalized Einstein tensor**. Here κ is a constant.

2 Conformal mappings of Riemannian manifolds

Let (M^n, g) and $(\overline{M}^n, \overline{g})$ be n-dimensional Riemannian manifolds with metric tensors g_{ij} and $\overline{g}_{ij}$ respectively. Both metrics are defined in a common coordinate system (x^i).

Definition *The correspondence between (M^n, g) and $(\overline{M}^n, \overline{g})$ is conformal, if the fundamental tensors g_{ij} and $\overline{g}_{ij}$ of two manifolds M^n and $\overline{M}^n$ are in the relation*

$$\overline{g}_{ij}(x) = e^{2\varphi(x)} g_{ij}(x), \tag{2}$$

where $\varphi(x)$ is a function of the x's.

Connections Γ_{ij}^k and $\overline{\Gamma}_{ij}^k$ compatible with the metrics g_{ij} and $\overline{g}_{ij}$ respectively must satisfy the equations

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k \varphi_j + \delta_j^k \varphi_i - \varphi^k g_{ij}, \tag{3}$$

where $\varphi_i = \frac{\partial \varphi}{\partial x^i}$. Also we have the equations [3], [6]

$$\overline{R}_{ijk}^h = R_{ijk}^h + \delta_k^h \varphi_{ij} - \delta_j^h \varphi_{ik} + g^{hl}(\varphi_{lk} g_{ij} - \varphi_{lj} g_{ik}) + (\delta_k^h g_{ij} - \delta_j^h g_{ik})\Delta_1 \varphi, \tag{4}$$

$$\overline{R}_{ij} = R_{ij} + (n-2)\varphi_{ij} + (\Delta_2 \varphi + (n-2)\Delta_1 \varphi) g_{ij}, \tag{5}$$

$$\overline{R} = e^{-2\varphi}(R + 2(n-1)\Delta_2 \varphi + (n-1)(n-2)\Delta_1 \varphi). \tag{6}$$

Here $\varphi_i = \partial_i \varphi$, $\Delta_1 \varphi = \varphi_i \varphi_j g^{ij}$, $\varphi_{ij} = \nabla_j \varphi_i - \varphi_i \varphi_j$. Also R_{ijk}^h and $\overline{R}_{ijk}^h$ are the Riemann tensors of the manifolds M^n and $\overline{M}^n$ correspondingly. We denote as $R_{ij} = R_{ij\alpha}^\alpha$ and $\overline{R}_{ij} = \overline{R}_{ij\alpha}^\alpha$ their Ricci tensors. Finally, $R = R_{ij} g^{ij}$ and $\overline{R} = \overline{R}_{ij} \overline{g}^{ij}$ are their scalar curvatures.

3 Conformal mappings preserving generalized Einstein tensor

It follows from (5) and (6) that the function φ must satisfy the system [3, p. 114]

$$\begin{aligned} \nabla_j \varphi_i = \varphi_i \varphi_j - \frac{1}{2} g_{ij} \Delta_1 \varphi + \frac{1}{n-2}\left(\overline{R}_{ij} - \frac{\overline{R}\overline{g}_{ij}}{2(n-1)}\right) \\ - \frac{1}{n-2}\left(R_{ij} - \frac{R g_{ij}}{2(n-1)}\right). \end{aligned} \tag{7}$$

It follows from (1) that the deformation of the generalized Einstein tensor can be written as

$$\overline{\mathfrak{E}}_{ij} - \mathfrak{E}_{ij} = \overline{R}_{ij} - \kappa \overline{R}\overline{g}_{ij} - R_{ij} + \kappa R g_{ij}. \tag{8}$$

Taking account of the preservation requirement, i. e. $\overline{\mathfrak{E}}_{ij} = \mathfrak{E}_{ij}$, from (8) we get

$$\overline{R}_{ij} - R_{ij} = \kappa \overline{R}\overline{g}_{ij} - \kappa R g_{ij}. \tag{9}$$

Since (9) holds we can rewrite (7) as

$$\nabla_j \varphi_i = \varphi_i \varphi_j - \frac{1}{2} g_{ij} \Delta_1 \varphi + \lambda \overline{R}\overline{g}_{ij} - \lambda R g_{ij}, \tag{10}$$

where $\lambda = \frac{2\kappa(n-1)-1}{2(n-1)(n-2)}$. Differentiating (10) covariantly with respect to x^k and the connection Γ of the manifold (M^n, g) we get

$$\begin{aligned}
\nabla_k\nabla_j\varphi_i =& \nabla_k\varphi_i\varphi_j + \varphi_i\nabla_k\varphi_j - \varphi_m g^{ml}\nabla_k\varphi_l g_{ij} + \lambda\partial_k\overline{R}\overline{g}_{ij} + 2\lambda\varphi_k\overline{R}\overline{g}_{ij} - \lambda\partial_k R g_{ij} = \\
=& \left(\varphi_i\varphi_k - \frac{1}{2}g_{ik}\Delta_1\varphi + \lambda\overline{R}\overline{g}_{ik} - \lambda R g_{ik}\right)\varphi_j + \varphi_i\nabla_k\varphi_j - \\
&- \varphi_m g^{ml}\left(\varphi_l\varphi_k - \frac{1}{2}g_{lk}\Delta_1\varphi + \lambda\overline{R}\overline{g}_{lk} - \lambda R g_{lk}\right)g_{ij} + \\
&+ \lambda\partial_k\overline{R}\overline{g}_{ij} + 2\lambda\varphi_k\overline{R}\overline{g}_{ij} - \lambda\partial_k R g_{ij} = \varphi_i\varphi_k\varphi_j - \frac{1}{2}g_{ik}\Delta_1\varphi\varphi_j + \\
&+ \lambda\varphi_j\overline{R}\overline{g}_{ik} - \lambda\varphi_j R g_{ik} + \varphi_i\nabla_k\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi\varphi_k + \lambda\varphi_k\overline{R}\overline{g}_{ij} + \\
&+ \lambda\varphi_k R g_{ij} + \lambda\partial_k\overline{R}\overline{g}_{ij} - \lambda\partial_k R g_{ij}
\end{aligned} \tag{11}$$

Alternating (11) in j and k and using the Ricci identity, we obtain

$$\varphi_\alpha R^\alpha_{ijk} = 2\lambda\left(\frac{1}{2}\partial_k\overline{R}\overline{g}_{ij} - \frac{1}{2}\partial_j\overline{R}\overline{g}_{ik} - \frac{1}{2}\partial_k R g_{ij} + \frac{1}{2}\partial_j R g_{ik} + R(\varphi_k g_{ij} - \varphi_j g_{ik})\right). \tag{12}$$

The condition (12) can be rewritten as

$$\varphi_\alpha R^\alpha_{ijk} - 2\varphi_\alpha\lambda R(\delta^\alpha_k g_{ij} - \delta^\alpha_j g_{ik}) = \lambda(\partial_k\overline{R}\overline{g}_{ij} - \partial_j\overline{R}\overline{g}_{ik} - \partial_k R g_{ij} + \partial_j R g_{ik}),$$

or

$$\varphi_\alpha Z^\alpha_{ijk} = \lambda(\partial_k\overline{R}\overline{g}_{ij} - \partial_j\overline{R}\overline{g}_{ik} - \partial_k R g_{ij} + \partial_j R g_{ik}), \tag{13}$$

where

$$Z^h_{ijk} \overset{\text{def}}{=} R^h_{ijk} - \frac{2\kappa(n-1)-1}{(n-1)(n-2)}R(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{14}$$

We can express φ_{ij} using (10)

$$\varphi_{ij} = -\frac{1}{2}g_{ij}\Delta_1\varphi + \lambda\overline{R}\overline{g}_{ij} - \lambda R g_{ij}.$$

If the expressions for φ_{ij} are substituted in (4), we find that

$$\overline{Z}^h_{ijk} = Z^h_{ijk},$$

where Z^h_{ijk} is defined in (14). Hence we obtain

Lemma *If manifolds (M^n, g) and $(\overline{M}^n, \overline{g})$, $(n > 3)$ are in the conformal correspondence and the mapping preserves the tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, $(\kappa = const)$, then the condition*

$$\varphi_\alpha Z^\alpha_{ijk} = \frac{2\kappa(n-1)-1}{2(n-1)(n-2)}(\partial_k\overline{R}\overline{g}_{ij} - \partial_j\overline{R}\overline{g}_{ik} - \partial_k R g_{ij} + \partial_j R g_{ik})$$

holds. Also, the tensor

$$Z^h_{ijk} = R^h_{ijk} - \frac{2\kappa(n-1)-1}{(n-1)(n-2)}R(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

is also preserved by the mapping.

Now we prove that for preserving of the generalized Einstein tensor more strong conditions must be satisfied. Let us contract the tensor Z^h_{ijk} for h and k.

$$Z^{\alpha}_{ij\alpha} = R^{\alpha}_{ij\alpha} - \frac{2\kappa(n-1)-1}{(n-1)(n-2)}R(\delta^{\alpha}_{\alpha}g_{ij} - \delta^{\alpha}_{j}g_{i\alpha}) = R_{ij} - \frac{2\kappa(n-1)-1}{(n-2)}Rg_{ij}.$$

Obviously, the tensor $Z^{\alpha}_{ij\alpha} = R_{ij} - \frac{2\kappa(n-1)-1}{(n-2)}Rg_{ij}$ formed by contraction also must be preserved by the mappings. Hence we obtain that if $\kappa \neq \frac{1}{n}$ the product Rg_{ij} also must be preserved, as a linear combination (constant coefficients) of the preserved tensors.

$$\mathfrak{E}_{ij} - Z^{\alpha}_{ij\alpha} = R_{ij} - \kappa Rg_{ij} - R_{ij} + \frac{2\kappa(n-1)-1}{(n-2)}Rg_{ij} = \frac{1-\kappa n}{n-2}Rg_{ij}.$$

Similarly, it can be shown that the conformal mapping also must preserve the Ricci tensor R_{ij}. Hence the system (7) can be written in the form

$$\nabla_j\varphi_i = \varphi_i\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi. \tag{15}$$

Let us differentiate (15) covariantly with respect to x^k and the connection Γ of the manifold (M^n, g). We get

$$\begin{aligned}\nabla_k\nabla_j\varphi_i =&\nabla_k\varphi_i\varphi_j + \varphi_i\nabla_k\varphi_j - \varphi_m g^{ml}\nabla_k\varphi_l g_{ij} =\\ =&\left(\varphi_i\varphi_k - \frac{1}{2}g_{ik}\Delta_1\varphi\right)\varphi_j + \varphi_i\nabla_k\varphi_j - \varphi_m g^{ml}\left(\varphi_l\varphi_k - \frac{1}{2}g_{lk}\Delta_1\varphi\right)g_{ij} =\\ =&\varphi_i\varphi_k\varphi_j - \frac{1}{2}g_{ik}\Delta_1\varphi\varphi_j + \varphi_i\nabla_k\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi\varphi_k.\end{aligned} \tag{16}$$

Alternating (16) in j and k and using the Ricci identity, we obtain

$$\varphi_\alpha R^{\alpha}_{ijk} = 0. \tag{17}$$

We can express φ_{ij} using (15)

$$\varphi_{ij} = \nabla_j\varphi_i - \varphi_i\varphi_j = -\frac{1}{2}g_{ij}\Delta_1\varphi,$$

and substitute in (4). Collecting the terms we have

$$\overline{R}^h_{ijk} = R^h_{ijk},$$

hence

Theorem 3.1 *If manifolds (M^n, g) and $(\overline{M}^n, \overline{g})$, $(n > 3)$ are in the conformal correspondence and the mapping preserves the tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa Rg_{ij}$, and $\kappa \neq \frac{1}{n}$, then the function φ generating the mapping, must satisfy the system of PDE's*

$$\nabla_j\varphi_i = \varphi_i\varphi_j - \frac{1}{2}g_{ij}\Delta_1\varphi,$$

whose conditions of integrability are

$$\varphi_\alpha R^\alpha_{ijk} = 0.$$

Also, the Riemann tensor R^h_{ijk}, the Ricci tensor R_{ij}, and the product Rg_{ij} are preserved by the mapping.

It's worth for noting that some of presented results have been obtained in [4], in particular, the equations (15) and (17).

Let us differentiate (10) covariantly with respect to x^k

$$\varphi_{\alpha,l} R^\alpha_{ijk} + \varphi_\alpha R^\alpha_{ijk,l} = 0.$$

Because of (15) and (17) it follows

$$-\frac{1}{2}\Delta_1\varphi R_{lijk} + \varphi_\alpha R^\alpha_{ijk,l} = 0. \tag{18}$$

If an explored manifold (M_n, g) is locally symmetric, then

$$R^h_{ijk,l} = 0.$$

Hence it follows from (18) that

$$\frac{1}{2}\Delta_1\varphi R_{lijk} = 0. \tag{19}$$

If a manifold (M_n, g) is recurrent, i. e. the covariant derivative of the Riemann tensor respect to x^l satisfies

$$R^h_{ijk,l} = \rho_l R^h_{ijk},$$

then because of (17), it follows from (18) that (19) holds.

Now, transvecting (17) and (18) with g^{ij} we get for Einstein manifilds

$$\frac{1}{2}\Delta_1\varphi R_{lk} = 0. \tag{20}$$

Thus, following (19) and (20), we have

Theorem 3.2 *Recurrent(in particular, symmetric) manifolds (M^n, g) $(n > 3)$, equipped by positive definite metric with the Riemann tensor which is not equal to zero, do not admit conformal mappings preserving the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, $(\kappa \neq \frac{1}{n})$. Also, Einstein manifolds equipped by positive definite metric, do not admit conformal mappings preserving the generalized Einstein tensor if they are not Ricci flat.*

Also for compact manifolds we have

Theorem 3.3 *Orientable compact manifolds (M^n, g) $(n > 2)$ equipped by positive definite metric, do not admit conformal mappings preserving the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, $(\kappa \neq \frac{1}{n})$.*

proof. Let a compact orientable manifold (M^n, g) $(n > 2)$ admits conformal mappings preserving the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, $(\kappa \neq \frac{1}{n})$. Then, the generating function φ must satisfy the equations (15):

$$\nabla_j \varphi_i = \varphi_i \varphi_j - \frac{1}{2} g_{ij} \Delta_1 \varphi.$$

Transvecting this with g^{ij}, we find

$$\nabla_i \varphi^i = -\frac{n-2}{2}\Delta_1 \varphi.$$

On the other hand, according to the Theorem of Green [9, p. 21]

$$\int\limits_{M^n} \nabla_i \varphi^i d\tau = 0,$$

where $d\tau$ is the volume element

$$d\tau = \sqrt{g} d\xi^1 \wedge d\xi^2 \wedge ... \wedge d\xi^n.$$

In this case we obtain

$$\frac{n-2}{2}\int\limits_{M^n} \Delta_1 \varphi d\tau = 0,$$

that is impossible for nontrivial conformal mappings of a manifold equipped by positive definite metric g. The theorem is proved.

4 Local structure of Riemann manifolds admitting conformal mapping preserving the generalized Einstein tensor

Let us multiply the both sides of (15) by $-e^{-\varphi}$ [5]. We get

$$-e^{-\varphi}\big(\nabla_j \varphi_i - \varphi_i \varphi_j\big) = e^{-\varphi}\frac{1}{2} g_{ij} \Delta_1 \varphi.$$

Then, putting $u = e^{-\varphi}$ from (15) we obtain

$$\nabla_j u_i = \frac{\Delta_1 u}{2u} g_{ij}. \tag{21}$$

Differentiating the multiplier $\frac{\Delta_1 u}{2u}$ we see that because of (21) any partial derivative of the multiplier is equal identically to zero

$$\frac{\partial}{\partial x^i}\big(\frac{\Delta_1 u}{2u}\big) = 0.$$

Thus, if a manifold admits nontrivial conformal mappings which preserve the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, $(\kappa \neq \frac{1}{n})$, then there exists a function u, that

$$\nabla_j u_i = c g_{ij}, \qquad c = const. \tag{22}$$

Suppose conversely that on a certain manifold (M^n, g) there exists a function satisfying (22). Multiplying the both sides of (22) by the vector u^i, we get

$$u^i \nabla_j u_i = c u_j,$$

or

$$\frac{1}{2}\frac{\partial}{\partial x^i}\big(\Delta_1 u\big) = c\frac{\partial u}{\partial x^i}.$$

It's obvious that

$$\frac{1}{2}\Delta_1 u = cu + c_2,$$

where c_2 is an arbitrary constant. We can express c using the equation and substitute the expression in (22)

$$\nabla_j u_i = \frac{\Delta_1 u}{2\left(u + \frac{c_2}{c}\right)} g_{ij}. \tag{23}$$

Since the constant c_2 is arbitrary, we can replace $u + \frac{c_2}{c}$ in (23) by $\tilde{u}$, satisfying the condition $\tilde{u} > 0$. Thus,

$$\nabla_j \tilde{u}_i = \frac{\Delta_1 \tilde{u}}{2\tilde{u}} g_{ij}. \tag{24}$$

Supposing that $\tilde{u} = e^{-\varphi}$, we derive that (15) holds. We suppose that $c > 0$, since the scalar field u can always be determined so that the conditions $c > 0$ and $u > 0$ hold. For example we can replace $u \longrightarrow u + C$, or $u \longrightarrow -u + C$, where C is a positive new constant. Obviously, the conditions of integrability of (22) are

$$u_t R^t_{ijk} = 0.$$

Thus we have

Theorem 4.1 *In order that a manifold (M^n, g) admit conformal mappings preserving the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, ($\kappa \neq \frac{1}{n}$) it is necessary and sufficient that there exist a scalar field u such that*

$$\nabla_j \nabla_i u = c g_{ij}, \qquad c = const \neq 0.$$

The problem of existence of such field was explored by P. A. Shirokov [8, p. 287]. Taking account of the Shirokov's results and using the theorem 4.1 , we have obtain

Theorem 4.2 *In order that a manifold (M^n, g) which is not Eclidean, admit conformal mappings preserving the generalized Einstein tensor $\mathfrak{E}_{ij} = R_{ij} - \kappa R g_{ij}$, ($\kappa \neq \frac{1}{n}$) it is necessary and sufficient that (M^n, g) must be irreducible and the metric g has the form*

$$ds^2 = (dx^1)^2 + (x^1)^2 h_{ts} dx^t dx^s, \qquad t, s = \overline{2, n}, \tag{25}$$

where coefficients h_{ts} do not depend on x^1. In that case only solution of the system (15) is the function generating the mapping

$$\varphi = ln \frac{1}{C_1 (x^1)^2 + C_2}, \tag{26}$$

where C_1 and C_2 are positive arbitrary constants.

If the metric g has the form (25), then the metric is said to define an Euclidean family of concentric spheres. As an example [8, p. 291] we can consider 3-dimensional space which contains such family. The metric g has the form

$$ds^2 = (dx^1)^2 + (x^1)^2 \left(g_{22}(dx^2)^2 + g_{33}(dx^3)^2\right),$$

and the function generating the conformal mapping preserving the generalized Einstein tensor is defined by the equation (26).

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