Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

PROPERTIES OF REDUCED MEET IDEALS IN LOWER BCK–SEMILATTICES

BORDBAR Hashem (IR), NOVÁK Michal (CZ), CRISTEA Irina (SI), JUN Young Bae (KR)

Abstract. In our paper we include some new results on lower BCK–semilattices, which are a special class of BCK-algebras. We concentrate on a specific c lass o f i deals o f lower BCK–semilattices. We make use of some recent results on weak closure operations on ideals of lower BCK–semilattices or relative annihilators in lower BCK–semilattices obtained by Jun et al.

Keywords: BCK–algebra, ideal, lower BCK–semilattice, relative annihilator

Mathematics subject classification: Primary 06F35, 06A12

1 Introduction

In our paper we include some new results in the area of BCK*–algebras*. This concept was introduced in 1966 by Iséki [16] and brought to a shape by Iséki and Tanaka in papers such as [17]. BCK–algebras are a special case of BCI–algebras, named so because their axioms work with certain combinators used in combinatory logic. For a collection of results on BCI– / BCK–algebras see [15, 20]; for the underlying theoretical concepts see e.g. papers by Bundler, Meyer or Hindley [6, 7, 14] and book [12] written by Curry and Feys.

Definition 1. *An algebra* $(X; *, 0)$ *of type* $(2, 0)$ *is called a* BCI–algebra *if it, for all* $x, y, z \in X$*, satisfies the following conditions:*

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) *simultaneous validity of* $x * y = 0$ *and* $y * x = 0$ *implies that* $x = y$*.*

In BCI–algebras we can define, for all $x, y \in X$, a relation "$\leq$", called a *BCI–ordering*, by setting

$$x \leq y \text{ whenever } x * y = 0. \tag{1}$$

It is easy to show that "$\leq$" is a partial ordering on X.

The nature of BCI–algebras is such that the operation "$*$" can often *intentionally* be neither associative nor commutative in the usual sense of the word. In the special case of *associative BCI–algebras*, i.e. BCI–algebras $(X; *, 0)$ such that, for all $x, y, z \in X$, there is $(x * y) * z = x * (y * z)$, we have that $x * y = y * x$ and $0 * x = x$ for all $x, y \in X$ (all these statements are in fact equivalent).

Definition 2. *A BCI–algebra is called a BCK*–algebra *if, for all $x \in X$, there is $0 * x = 0$. A BCK–algebra is called* commutative *if, for all $x, y \in X$ there is $x * (x * y) = y * (y * x)$. A BCK–algebra is called a* lower BCK–semilattice *if $(X, \leq)$, where "$\leq$" is a BCI–ordering, is a lower semilattice.*

Thus, lower BCK–semilattices are a special – yet distinct and important – class of lower semilattices. *Bounded* BCI/BCK–algebras are such BCI/BCK–algebras that have the greatest element, which is usually denoted by 1. In a bounded commutative BCK–algebra the least upper bound of an arbitrary pair of elements $x, y \in X$ satisfies

$$x \vee y = 1 * ((1 * x) \wedge (1 * y)),$$

which means that, in this case, $(X, \leq)$ is a *distributive lattice*. Notice that bounded commutative BCK–algebras are in fact *MV–algebras* introduced by Chang [11].

In our paper we focus on *lower BCK–semilattices* and their *ideals*. The concept of an *ideal* is one of the corner stones of the theory of BCI– / BCK–algebras (see [20]).

Definition 3. *A subset A of a BCK/BCI-algebra X is called an* ideal *of X if, for all $x, y \in X$, there is*

$$0 \in A, \tag{2}$$
$$x * y \in A, y \in A \Rightarrow \ x \in A. \tag{3}$$

Note that – obviously – every ideal A of a BCK/BCI–algebra X satisfies, for all $x, y \in X$, the following implication:

$$x \leq y, \ y \in A \ \Rightarrow \ x \in A. \tag{4}$$

The following terminology is standard for numerous algebraic concepts.

Definition 4. *For any subset A of X, the* ideal generated by A *is defined to be the intersection of all ideals of X containing A. We denote it $\langle A \rangle$. If A is finite, we say that $\langle A \rangle$ is a* finitely generated ideal *of X.*

In what follows we by $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ mean the set of all ideals of X and the set of all finitely generated ideals of X, respectively. By $(X, \wedge, \leq)$ we mean, unless specified otherwise, a *lower BCK–semilattice*.

2 Preliminaries

In Section 3 we make use of some results on *weak closure operations* on ideals of BCK–algebras or *relative annihilators* in lower BCK–semilattices obtained by Jun et al. [3, 4, 5]. Notice that closure operations or (relative) annihilators are studied for various types of combinatory logic algebras – see e.g. Chajda and Rachůnek [10]. For some results concerning lattices within the theory of BCK–algebras see e.g. papers by Chajda and Länger such as [8, 9].

For the proofs of results included in this section see [4, 5].

Definition 5. *For any nonempty subsets A and B of $(X, \wedge, \leq)$, we denote*

$$A \wedge B := \{a \wedge b \mid a \in A, b \in B\}, \tag{5}$$

which is called the meet set *of X generated by A and B. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.*

The following example shows that the set $A \wedge B$ need not always be an ideal of X if A, B are *arbitrary* subsets of X.

Example 1. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

X has six ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 2, 4\}$ and $A_5 = X$. For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$A \wedge B = \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\},$$

which is not an ideal of X.

Therefore, we provide conditions for the meet set $A \wedge B$ to be an ideal.

Theorem 1. *If A and B are ideals of X, then so is their meet set $A \wedge B$.*

Proposition 1. *If A, B and C are ideals of X, then*

(i) $A \wedge 0 = 0$.

(ii) $A \wedge B = A \cap B$.

(iii) $A \wedge B) \wedge C = A \wedge (B \wedge C) = \{a \wedge b \wedge c \mid a \in A, b \in B, c \in C\}$.

Obviously, $A \wedge B = B \wedge A$ for an arbitrary pair of nonempty subsets A and B of X. For any nonempty subsets A, B and C of X, we have

$$A \subseteq B,\ A \subseteq C \ \Rightarrow\ A \subseteq B \wedge C. \tag{6}$$

The following example shows that there exist subsets A, B and C of X such that $A \subseteq B$ and $A \subseteq C$, but $B \wedge C \nsubseteq A$.

Example 2. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the one in Example 1).*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of X, we have

$$B \wedge C = \{b \wedge c \mid b \in B,\ c \in C\} = \{0, 1, 2\} \nsubseteq \{0, 1\} = A.$$

Definition 6. [4] *For any nonempty subsets A and B of X, we define a set*

$$(A :_\wedge B) := \{x \in X \mid x \wedge B \subseteq A\} \tag{7}$$

which is called the relative annihilator *of B with respect to A.*

Given a lower BCK-semilattice X, note that if $A = \{0\}$, then

$$\begin{aligned}(\{0\} :_\wedge B) &= \{x \in X \mid x \wedge B \subseteq \{0\}\} \\ &= \{x \in X \mid x \wedge b = 0,\ \forall b \in B\} \\ &= B^*\end{aligned} \tag{8}$$

which is the annihilator of B (see Huang [15]). Hence the concept of the *relative annihilator of B with respect to A* is a generalization of the concept of the *annihilator of B*.

Proposition 2. [4] *For any nonempty subsets A and B of X, we have*

(i) If A is an ideal of X, then $A \subseteq (A :_\wedge B)$.

(ii) If $B_1 \subseteq B_2$ in X, then $(A :_\wedge B_2) \subseteq (A :_\wedge B_1)$.

Lemma 1. [4] *If A and B are ideals of X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

For any nonempty subsets A and B of X, let us now consider the set

$$F := \{x \in X \mid x \wedge B \subseteq A \wedge B\} \tag{9}$$

where $x \wedge B$ and $A \wedge B$ are meet sets. The following example shows that such a set need not be an ideal.

Example 3. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the one in Example 2).*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

We have five ideals of X: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$. For subsets $A = \{2, 3\}$ and $B = \{4\}$ of X, we have

$$\{x \in X \mid x \wedge B \subseteq A \wedge B\} = \{1, 2, 3\},$$

which is not an ideal of X.

3 Some new results on ideals of lower BCK–semilattices

First of all, we provide conditions for the set $F = \{x \in X \mid x \wedge B \subseteq A \wedge B\}$ to be an ideal.

Theorem 2. *If A and B are ideals of a lower BCK-semilattice X, then the set*

$$F = \{x \in X \mid x \wedge B \subseteq A \wedge B\}$$

is an ideal of X.

Proof. Since A and B are ideals of X, using Theorem 1 shows that $A \wedge B$ is an ideal. So $0 \in A \wedge B$. Also, $0 \wedge B = 0 \in A \wedge B$. Hence, $0 \in F$.

Let $x, y \in X$ be such that $x * y \in F$ and $y \in F$. Then $(x * y) \wedge B \subseteq A \wedge B$ and $y \wedge B \subseteq A \wedge B$, that is,

$$(x * y) \wedge b \in A \wedge B \tag{10}$$

and

$$y \wedge b \in A \wedge B \tag{11}$$

for all $b \in B$. Since $b \in B$, we have $\langle b \rangle \subseteq B$. It follows that $y \wedge \langle b \rangle \subseteq y \wedge B$ and $(x * y) \wedge \langle b \rangle \subseteq (x * y) \wedge B$. Thus, $y \wedge \langle b \rangle \subseteq A \wedge B$ and $(x * y) \wedge \langle b \rangle \subseteq A \wedge B$. It means that,

$$y \in (A \wedge B :_{\wedge} \langle b \rangle) \tag{12}$$

and

$$x * y \in (A \wedge B :_{\wedge} \langle b \rangle) \tag{13}$$

Using Lemma 1, we conclude that $(A \wedge B :_{\wedge} \langle b \rangle)$ is an ideal of X. So (12), (13) and (3) show that

$$x \in (A \wedge B :_{\wedge} \langle b \rangle)$$

Hence, or all $b \in B$, $x \wedge b \in x \wedge \langle b \rangle \subseteq A \wedge B$. Therefore,

$$x \wedge B \subseteq A \wedge B,$$

which means that $x \in F$. □

In the following text we are going to investigate properties of such ideals. The name, which we choose for them, stresses the fact that the set A is reduced to a one element set.

Definition 7. *For two ideals A and B of X, the ideal*

$$\{x \in X \mid x \wedge B \subseteq A \wedge B\}$$

*is called A-*reduced meet ideal based on B *and denoted by $\frac{A \wedge B}{B}$.*

The following example shows that the converse of Theorem 2 is not true in general, which means that there exist subsets A and B of X which are not ideals, but $\frac{A \wedge B}{B}$ is an ideal of X.

Example 4. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the ones in Example 2 or Example 3).*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

We have four ideals of X: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2, 4\}$ and $A_3 = X$. For subsets $A = \{0, 1\}$ and $B = \{1, 2\}$ of X which are not ideals, we have

$$\frac{A \wedge B}{B} = \{0, 1, 3\},$$

which is the ideal A_1 of X.

Proposition 3. *For any ideals A and B of X, we have*

(i) $A \subseteq \frac{A \wedge B}{B}$ *and* $B \subseteq \frac{A \wedge \frac{A \wedge B}{B}}{\frac{A \wedge B}{B}}$.

(ii) $\frac{A \wedge X}{X} = A$ *and* $\frac{A \wedge A}{A} = X$.

(iii) $\frac{A \wedge B}{B} = X \ \Leftrightarrow \ B \subseteq A$.

Proof. (i) Suppose an arbitrary $x \in A$. Then $x \wedge B \subseteq A \wedge B$ for all $B \in \mathcal{I}(X)$. Thus $x \in \frac{A \wedge B}{B}$, and so $A \subseteq \frac{A \wedge B}{B}$.

Let $x \in B$ be an arbitrary element. For any arbitrary element $y \in \frac{A \wedge B}{B}$, we have $y \wedge B \subseteq A \wedge B$. Since $x \in B$, it follows that $y \wedge x \in A \wedge B$. Thus,

$$x \wedge \frac{A \wedge B}{B} \subseteq A \wedge B.$$

Besides,

$$A \wedge B \subseteq A = A \wedge \frac{A \wedge B}{B}.$$

Therefore, $x \wedge \frac{A\wedge B}{B} \subseteq A \wedge \frac{A\wedge B}{B}$, implying that

$$x \in \frac{A \wedge \frac{A\wedge B}{B}}{\frac{A\wedge B}{B}}.$$

(ii) Let $x \in A$. Then $x\wedge X \subseteq A\wedge X$, and so $x \in \frac{A\wedge X}{X}$. Conversely, let $x \in \frac{A\wedge X}{X}$. Then $x\wedge X \subseteq A\wedge X$. Since $A \wedge X = A$ and $x \wedge x = x$, we have $x \in A$. Therefore,

$$\frac{A \wedge X}{X} = A.$$

Clearly,

$$\frac{A \wedge A}{A} = \{x \in X \mid x \wedge A \subseteq A \wedge A\} = X.$$

(iii) Suppose that $\frac{A\wedge B}{B} = X$. Let b be an arbitrary element of B. Since $B \subseteq X$, clearly we have $b \wedge B \subseteq A \wedge B$, and so

$$b = b \wedge b \in A \wedge B.$$

Thus $B \subseteq A \wedge B$. Also, we always have $A \wedge B \subseteq B$. Therefore, $A \wedge B = B$ which means that $B \subseteq A$.

Conversely, suppose that $B \subseteq A$. Let $x \in X$ and $b \in B$ be an arbitrary element. Then $x \wedge b \leq b$, and thus by using (4), $x \wedge b \in B \subseteq A$, i.e. $x \wedge b \in A \wedge B$ and $x \in \frac{A\wedge B}{B}$. Thus $X \subseteq \frac{A\wedge B}{B}$, and so $X = \frac{A\wedge B}{B}$. □

The following example shows that the converse of Proposition 3 (i), is not true in general, meaning that there exist ideals A and B of X such that $\frac{A\wedge B}{B} \nsubseteq A$ and $\frac{A\wedge \frac{A\wedge B}{B}}{\frac{A\wedge B}{B}} \nsubseteq B$.

Example 5. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (again different from all the above examples).*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

We have six ideals of X: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$. For ideals $A = A_2 = \{0, 1, 2\}$ and $B = A_1 = \{0, 1\}$ of X we have

$$\frac{A \wedge B}{B} = A_5,$$

and clearly,

$$\frac{A \wedge B}{B} \nsubseteq A.$$

Also,

$$A \wedge \frac{A \wedge B}{B} = A \wedge A_5 = A.$$

Thus,

$$\{x \in X \mid x \wedge \frac{A \wedge B}{B} \subseteq A \wedge \frac{A \wedge B}{B}\} = \{x \in X \mid x \wedge A_5 \subseteq A\} = \{0, 1, 2\},$$

and hence,

$$\frac{A \wedge \frac{A \wedge B}{B}}{\frac{A \wedge B}{B}} \nsubseteq B.$$

Proposition 4. *For any ideals A and B of X such that $A \subseteq B$, we have*

$$\frac{A \wedge B}{B} \cap B = A.$$

Proof. Let $x \in A$. Then clearly $x \in B$ and $x \wedge B \subseteq A \wedge B$. Thus

$$x \in \frac{A \wedge B}{B} \cap B.$$

Now suppose that $x \in \frac{A \wedge B}{B} \cap B$. Then $x \in B$ and $x \in \frac{A \wedge B}{B}$. So $x \wedge B \subseteq A \wedge B$. Since $A \subseteq B$ and $x \in B$, we have $A \wedge B = A$ and

$$x = x \wedge x \in A \wedge B = A.$$

Therefore, $x \in A$ and

$$\frac{A \wedge B}{B} \cap B = A.$$

□

Proposition 5. *Let B be an ideal of X. For any family $\{A_\lambda \mid \lambda \in \Lambda\}$ of ideals, we have*

$$\left(\frac{\bigcap_{\lambda \in \Lambda} A_\lambda \wedge B}{B}\right) = \bigcap_{\lambda \in \Lambda} \left(\frac{A_\lambda \wedge B}{B}\right). \tag{14}$$

Proof. Let $x \in \left(\frac{\bigcap_{\lambda \in \Lambda} A_\lambda \wedge B}{B}\right)$. Then $x \wedge B \subseteq \bigcap_{\lambda \in \Lambda} A_\lambda \wedge B$. So for an arbitrary element $y \in B$, $x \wedge y \in \bigcap_{\lambda \in \Lambda} A_\lambda \wedge B$. Hence there exist $a \in \bigcap_{\lambda \in \Lambda} A_\lambda$ and $b \in B$ such that $x \wedge y = a \wedge b$. Thus, for every $\lambda \in \Lambda$, $a \wedge b \in A_\lambda \wedge B$ and so $x \wedge y \in A_\lambda \wedge B$. Since y is an arbitrary element of B, we have $x \wedge B \subseteq A_\lambda \wedge B$. Hence, for every $\lambda \in \Lambda$,

$$x \in \left(\frac{A_\lambda \wedge B}{B}\right)$$

which means that

$$x \in \bigcap_{\lambda \in \Lambda} \left(\frac{A_\lambda \wedge B}{B}\right).$$

Therefore,

$$\left(\frac{\bigcap_{\lambda \in \Lambda} A_\lambda \wedge B}{B}\right) \subseteq \bigcap_{\lambda \in \Lambda} \left(\frac{A_\lambda \wedge B}{B}\right).$$

Now suppose that $x \in \bigcap\limits_{\lambda\in\Lambda} \left(\frac{A_\lambda \wedge B}{B}\right)$. Then for every $\lambda \in \Lambda$, $x \in \frac{A_\lambda \wedge B}{B}$ and so $x \wedge B \subseteq A_\lambda \wedge B$. For an arbitrary element $y \in B$ and every $\lambda \in \Lambda$, $x \wedge y \in A_\lambda \wedge B$. So for $\lambda_1 \in \Lambda$, there exist $a_{\lambda_1} \in A_{\lambda_1}$ and $b_1 \in B$ such that $x \wedge y = a_{\lambda_1} \wedge b_1$. Similarly, for $\lambda_i \in \Lambda$ $(2 \leq i \leq n)$, there exist $a_{\lambda_i} \in A_{\lambda_i}$ and $b_i \in B$ such that $x \wedge y = a_{\lambda_i} \wedge b_i$. Now we have

$$\begin{aligned} x \wedge y =& (x \wedge y) \wedge (x \wedge y) \wedge \ldots \wedge (x \wedge y) \\ &= (a_{\lambda_1} \wedge b_1) \wedge (a_{\lambda_2} \wedge b_2) \wedge \ldots \wedge (a_{\lambda_n} \wedge b_n) \\ &= (a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n}) \wedge (b_1 \wedge b_2 \ldots \wedge b_n) \end{aligned}$$

Clearly, $a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n} \in \bigwedge\limits_{\lambda\in\Lambda} A_\lambda$ and $b_1 \wedge b_2 \ldots \wedge b_n \in B$. Using Proposition 1, we conclude that

$$a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n} \in \bigcap_{\lambda\in\Lambda} A_\lambda.$$

Thus, $x \wedge y \in \bigcap\limits_{\lambda\in\Lambda} A_\lambda \wedge B$. Since y is an arbitrary element of X, we have

$$x \in \left(\frac{\bigcap\limits_{\lambda\in\Lambda} A_\lambda \wedge B}{B}\right).$$

Therefore,

$$\left(\frac{\bigcap\limits_{\lambda\in\Lambda} A_\lambda \wedge B}{B}\right) = \bigcap_{\lambda\in\Lambda} \left(\frac{A_\lambda \wedge B}{B}\right).$$

$\square$

Theorem 3. *For any ideals A and B of X, we have*

$$A \wedge B \subseteq \frac{A \wedge B}{B} \subseteq (A :_\wedge B). \tag{15}$$

Moreover, if $A \subseteq B$, then

$$\frac{A \wedge B}{B} = (A :_\wedge B). \tag{16}$$

Proof. Let $x \in A \wedge B$ be an arbitrary element. Then there exist $a \in A$ and $b \in B$ such that $x = a \wedge b$. For an arbitrary element $b' \in B$ we have

$$x \wedge b' = (a \wedge b) \wedge b' = a \wedge (b \wedge b').$$

Since $b \wedge b' \leq b$ and $b \in B$, using (4) we conclude that $b \wedge b' \in B$. Thus, $a \wedge (b \wedge b') \in A \wedge B$. Therefore, $A \wedge B \subseteq \frac{A \wedge B}{B}$.

Now let $x \in \frac{A \wedge B}{B}$. Then for every element $b \in B$, $x \wedge b \in A \wedge B$. Since $A \wedge B \leq A$, we have $x \wedge b \in A$. Hence $x \wedge B \subseteq A$ and $x \in (A :_\wedge B)$. Therefore, $\frac{A \wedge B}{B} \subseteq (A :_\wedge B)$.

If $A \subseteq B$, then $A \wedge B = A$ and equation (16) is clear. $\square$

Theorem 4. *For ideals A, B_1 and B_2 of X, if $B_1 \subseteq B_2$, then*

$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} \subseteq \frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)}. \tag{17}$$

Proof. Suppose that $B_1 \subseteq B_2$ and $x \in \frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)}$. Then

$$x \wedge (A :_\wedge B_1) \subseteq A \wedge (A :_\wedge B_1).$$

Using Proposition 2, we have

$$A \wedge (A :_\wedge B_1) = A = A \wedge (A :_\wedge B_2),$$
$$(A :_\wedge B_2) \subseteq (A :_\wedge B_1).$$

Thereby,

$$x \wedge (A :_\wedge B_2) \subseteq x \wedge (A :_\wedge B_1) \subseteq A \wedge (A :_\wedge B_1) = A = A \wedge (A :_\wedge B_2).$$

Therefore, $x \wedge (A :_\wedge B_2) \subseteq A \wedge (A :_\wedge B_2)$ and thus

$$x \in \frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)}.$$

□

The following example shows that the converse of inclusion (17) is not true in general, which means that there exist ideals A, B_1 and B_2 of X such that $B_1 \subseteq B_2$, but

$$\frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} \not\subseteq \frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)}$$

Example 6. *Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For ideals $A = \{0, 1\}$, $B_1 = \{0, 3\}$ and $B_2 = \{0, 2, 3\}$ of X, we have

$$(A :_\wedge B_1) = \{x \in X \mid x \wedge B_1 \subseteq A\} = \{0, 1, 2, 4\},$$

and

$$(A :_\wedge B_2) = \{x \in X \mid x \wedge B_2 \subseteq A\} = \{0, 1, 4\}.$$

Thus,

$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} = \{x \in X \mid x \wedge (A :_\wedge B_1) \subseteq A \wedge (A :_\wedge B_1)\} = \{0, 1\},$$

and

$$\frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} = \{x \in X \mid x \wedge (A :_\wedge B_2) \subseteq A \wedge (A :_\wedge B_2)\} = \{0, 1, 2, 3\}.$$

Clearly,
$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} \subseteq \frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)}.$$
But
$$\frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} \not\subseteq \frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)}.$$

Proposition 6. *For ideals A, B_1 and B_2 of X, we have*
$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} \cap \frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} \subseteq \frac{A \wedge (A :_\wedge \langle B_1 \cup B_2 \rangle)}{(A :_\wedge \langle B_1 \cup B_2 \rangle)}. \tag{18}$$

Proof. Since $B_1 \subseteq \langle B_1 \cup B_2 \rangle$, using Theorem 4, we conclude that
$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} \subseteq \frac{A \wedge (A :_\wedge \langle B_1 \cup B_2 \rangle)}{(A :_\wedge \langle B_1 \cup B_2 \rangle)}.$$
Similarly, we have
$$\frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} \subseteq \frac{A \wedge (A :_\wedge \langle B_1 \cup B_2 \rangle)}{(A :_\wedge \langle B_1 \cup B_2 \rangle)}.$$
Therefore,
$$\frac{A \wedge (A :_\wedge B_1)}{(A :_\wedge B_1)} \cap \frac{A \wedge (A :_\wedge B_2)}{(A :_\wedge B_2)} \subseteq \frac{A \wedge (A :_\wedge \langle B_1 \cup B_2 \rangle)}{(A :_\wedge \langle B_1 \cup B_2 \rangle)}.$$
□

4 Future work

The roots of BCI–, BCK–, MV– and other types of algebras lie in combinatory logic and information sciences. Notice that recently Jun and Song [18] and Flaut [13] linked BCK–algebras to block codes used in channel encoding in earlier mobile communication systems. Matrices of block codes are studied in Saeid et al. [21]. Since, in a special case, we obtain lattices (or semilattices), whenever multivalued aspects are employed we can make use of results of the hyperstructure theory and concepts such as e.g. EL–hyperstructures (see e.g. Křehlík and Novák [19] where sets of matrices and lattices are studied) or even classical results of Varlet [22] who provided the link between distributive lattices and hyperstructure theory. In our future work we shall concentrate on finding links between our results and the above mentioned concepts, or those connected with closure operations on BCK–algebras [1, 2].

Acknowledgement

Michal Novák was supported by the project FEKT-S-17-4225 of Brno University of Technology. Irina Cristea acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1 – 0285).

References

[1] BORDBAR, H., ZAHEDI, M. M.: A finite type of closure operations on BCK-algebra, *Appl. Math. Inf. Sci. Lett.*, **4**(2) (2016), pp. 1–9.

[2] BORDBAR, H., ZAHEDI, M. M.: Semi-prime closure operations on BCK-algebra, *Commun. Korean Math. Soc.*, **30**(5) (2015), pp. 385–402.

[3] BORDBAR, H., ZAHEDI, M. M., AHN, S. S., JUN, Y. B.: Weak closure operations on ideals of BCK-algebras, *J. Comput. Anal. Appl.* (in press).

[4] BORDBAR, H., ZAHEDI, M. M., JUN, Y. B.: Relative annihilators in lower BCK-semilattices, *Math. Sci. Lett.*, **6**(2), (2017), pp. 1–7.

[5] BORDBAR, H., ZAHEDI, M. M., AHN, S. S., JUN, Y. B.: Semiring structures based on meet and plus ideals in lower BCK-semilattices, *J. Comput. Anal. Appl.*, (2017). (in press).

[6] BUNDLER, M. W.: Corrections to some results for **BCK** logics and algebras, *Logique et Analyse*, **31**(121/122) (1988), pp. 115–122.

[7] BUNDLER, M. W., MEYER, R. K.: A result for combinators, **BCK** logics and **BCK**–algebras, *Logique et Analyse*, **109** (1985), pp. 33–40.

[8] CHAJDA, I.: A structure of BCI–algebras, *Int. J. Theor. Phys.*, **53**(10) (2014), pp. 3391–3396.

[9] CHAJDA, I., LÄNGER, H.: Weak Lattices, *Ital. J. Pure Appl. Math.*, **30** (2013), pp. 125–140.

[10] CHAJDA, I., RACHŮNEK: J: Annihilators in normal autometrized algebras, *Czech. Math. J.*, **51**(1) (2001), pp. 111–120.

[11] CHANG, C. C.: Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.*, **88** (1958), pp. 467-490.

[12] CURRY, H. B., FEYS, R.: Combinatory Logic, North-Holland, Amsterdam, 1958.

[13] FLAUT, C.: BCK–algebras arising from block codes, *J. Intell. Fuzzy Syst.*, **28**(4) (2015), pp. 1829–1833.

[14] HINDLEY, R.: The principal type-scheme of an object in combinatory logic, *Trans. Amer. Math. Soc.*, **146** (1969), pp. 29–60.

[15] HUANG, Y.: BCI-algebra, Beijing: Science Press, 2006.

[16] ISÉKI, K: An algebra related with a propositional calculus, *Proc. Japan. Acad.*, **42** (1966), pp. 26–29.

[17] ISÉKI, K: An introduction to the theory of BCK–algebras, *Math. Japon.*, **23** (1978), pp. 1–26.

[18] JUN, Y. B., SONG, S. Z.: Codes based on BCK–algebras, *Inform. Sci.*, **181** (2015), pp. 5102–5109.

[19] KŘEHLÍK, Š., NOVÁK, M: From lattices to H_v–matrices, *An. Şt. Univ. Ovidius Constanţa*, **24**(3) (2016), pp. 209–222.

[20] MENG, J., JUN, Y. B., BCK-algebras, Seoul: Kyung Moon Sa Co. 1994.

[21] SAEID, A. E. et al.: Some connections between BCK–algebras and n–ary block codes, *Soft Comput.* (2017) (online ready).

[22] VARLET, J. V.: Remarks on distributive lattices, *Bull. de l'Acad. Polonnaise des Sciences, Serie des Sciences Math., Astr. et Phys.*, **23** (1975), pp. 1143–1147.

[23] VASSILEV, J. C.: Structure on the Set of Closure Operations of a Commutative Ring, *J. Algebr.*, **321**(10) (2009), pp. 2737-2753.

Current address

Bordbar Hashem, Ph.D.
Faculty of Mathematics, Statistics and Computer Science
Shahid Beheshti University
Tehran, Iran
E-mail: bordbar.amirh@gmail.com

Novák Michal, RNDr. Ph.D.
Faculty of Electrical Engineering and Communication
Brno University of Technology
Technická 8, 616 00 Brno, Czech Republic
E-mail: novakm@feec.vutbr.cz

Cristea Irina, prof. dr.
Center for Information Technologies and Applied Mathematics
University of Nova Gorica
Vipavska 13, SI-5000 Nova Gorica, Slovenia
E-mail: irinacri@yahoo.co.uk

Jun Young Bae, Prof. Emeritus
Department of Mathematics Education (and RINS)
Gyeongsang National University
Jinju 52828, Korea
E-mail: skywine@gmail.com