Abstract. In our paper we include some new results on lower $BCK$–semilattices, which are a special class of $BCK$-algebras. We concentrate on a specific class of ideals of lower $BCK$–semilattices. We make use of some recent results on weak closure operations on ideals of lower $BCK$–semilattices or relative annihilators in lower $BCK$–semilattices obtained by Jun et al.

Keywords: $BCK$–algebra, ideal, lower $BCK$–semilattice, relative annihilator

Mathematics subject classification: Primary 06F35, 06A12

1 Introduction

In our paper we include some new results in the area of $BCK$–algebras. This concept was introduced in 1966 by Iséki [16] and brought to a shape by Iséki and Tanaka in papers such as [17]. $BCK$–algebras are a special case of $BCI$–algebras, named so because their axioms work with certain combinators used in combinatory logic. For a collection of results on $BCI$–$BCK$–algebras see [15, 20]; for the underlying theoretical concepts see e.g. papers by Bundler, Meyer or Hindley [6, 7, 14] and book [12] written by Curry and Feys.

Definition 1. An algebra $(X; *, 0)$ of type $(2, 0)$ is called a $BCI$–algebra if it, for all $x, y, z \in X$, satisfies the following conditions:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) simultaneous validity of $x * y = 0$ and $y * x = 0$ implies that $x = y$. 
In $BCI$–algebras we can define, for all $x, y \in X$, a relation “$\leq$”, called a $BCI$–ordering, by setting

$$x \leq y \text{ whenever } x \ast y = 0. \quad (1)$$

It is easy to show that “$\leq$” is a partial ordering on $X$.

The nature of $BCI$–algebras is such that the operation “$\ast$” can often intentionally be neither associative nor commutative in the usual sense of the word. In the special case of associative $BCI$–algebras, i.e. $BCI$–algebras $(X; \ast, 0)$ such that, for all $x, y, z \in X$, there is $(x \ast y) \ast z = x \ast (y \ast z)$, we have that $x \ast y = y \ast x$ and $0 \ast x = x$ for all $x, y \in X$ (all these statements are in fact equivalent).

**Definition 2.** A $BCI$–algebra is called a $BCK$–algebra if, for all $x \in X$, there is $0 \ast x = 0$. A $BCK$–algebra is called commutative if, for all $x, y \in X$ there is $x \ast (x \ast y) = y \ast (y \ast x)$. A $BCK$–algebra is called a lower $BCK$–semilattice if $(X, \leq)$, where “$\leq$” is a $BCI$–ordering, is a lower semilattice.

Thus, lower $BCK$–semilattices are a special – yet distinct and important – class of lower semilattices. Bounded $BCI/BCK$–algebras are such $BCI/BCK$–algebras that have the greatest element, which is usually denoted by 1. In a bounded commutative $BCK$–algebra the least upper bound of an arbitrary pair of elements $x, y \in X$ satisfies

$$x \lor y = 1 \ast ((1 \ast x) \land (1 \ast y)),$$

which means that, in this case, $(X, \leq)$ is a distributive lattice. Notice that bounded commutative $BCK$–algebras are in fact $MV$–algebras introduced by Chang [11].

In our paper we focus on lower $BCK$–semilattices and their ideals. The concept of an ideal is one of the cornerstones of the theory of $BCI$–/$BCK$–algebras (see [20]).

**Definition 3.** A subset $A$ of a $BCK/BCI$-algebra $X$ is called an ideal of $X$ if, for all $x, y \in X$, there is

$$0 \in A,$$  \hspace{1cm} (2)

$$x \ast y \in A, y \in A \Rightarrow x \in A. \quad (3)$$

Note that – obviously – every ideal $A$ of a $BCK/BCI$–algebra $X$ satisfies, for all $x, y \in X$, the following implication:

$$x \leq y, y \in A \Rightarrow x \in A. \quad (4)$$

The following terminology is standard for numerous algebraic concepts.

**Definition 4.** For any subset $A$ of $X$, the ideal generated by $A$ is defined to be the intersection of all ideals of $X$ containing $A$. We denote it $\langle A \rangle$. If $A$ is finite, we say that $\langle A \rangle$ is a finitely generated ideal of $X$.

In what follows we by $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ mean the set of all ideals of $X$ and the set of all finitely generated ideals of $X$, respectively. By $(X, \land, \leq)$ we mean, unless specified otherwise, a lower $BCK$–semilattice.
2 Preliminaries

In Section 3 we make use of some results on weak closure operations on ideals of $BCK$–algebras or relative annihilators in lower $BCK$–semilattices obtained by Jun et al. [3, 4, 5]. Notice that closure operations or (relative) annihilators are studied for various types of combinatory logic algebras – see e.g. Chajda and Rachůnek [10]. For some results concerning lattices within the theory of $BCK$–algebras see e.g. papers by Chajda and Länger such as [8, 9].

For the proofs of results included in this section see [4, 5].

Definition 5. For any nonempty subsets $A$ and $B$ of $(X, \land, \leq)$, we denote

$$A \land B := \{ a \land b \mid a \in A, b \in B \},$$

which is called the meet set of $X$ generated by $A$ and $B$. If $A = \{ a \}$, then $\{a\} \land B$ is denoted by $a \land B$. Also, if $B = \{ b \}$, then $A \land \{b\}$ is denoted by $A \land b$.

The following example shows that the set $A \land B$ need not always be an ideal of $X$ if $A$, $B$ are arbitrary subsets of $X$.

Example 1. Consider the lower $BCK$-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

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$X$ has six ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 2, 4\}$ and $A_5 = X$. For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have

$$A \land B = \{ a \land b \mid a \in A, b \in B \} = \{0, 1, 2\},$$

which is not an ideal of $X$.

Therefore, we provide conditions for the meet set $A \land B$ to be an ideal.

Theorem 1. If $A$ and $B$ are ideals of $X$, then so is their meet set $A \land B$.

Proposition 1. If $A$, $B$ and $C$ are ideals of $X$, then

(i) $A \land 0 = 0$.

(ii) $A \land B = A \cap B$.

(iii) $A \land (B \land C) = A \land (B \land C) = \{ a \land b \land c \mid a \in A, b \in B, c \in C \}$. 

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Obviously, $A \land B = B \land A$ for an arbitrary pair of nonempty subsets $A$ and $B$ of $X$. For any nonempty subsets $A$, $B$ and $C$ of $X$, we have

$$A \subseteq B, \ A \subseteq C \Rightarrow A \subseteq B \land C. \quad (6)$$

The following example shows that there exist subsets $A$, $B$ and $C$ of $X$ such that $A \subseteq B$ and $A \subseteq C$, but $B \land C \not\subseteq A$.

**Example 2.** Consider the lower $BCK$-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the one in Example 1).

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For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of $X$, we have

$$B \land C = \{b \land c \mid b \in B, \ c \in C\} = \{0, 1, 2\} \not\subseteq \{0, 1\} = A.$$

**Definition 6.** [4] For any nonempty subsets $A$ and $B$ of $X$, we define a set

$$(A : \land B) := \{x \in X \mid x \land B \subseteq A\} \quad (7)$$

which is called the relative annihilator of $B$ with respect to $A$.

Given a lower $BCK$-semilattice $X$, note that if $A = \{0\}$, then

$$(\{0\} : \land B) = \{x \in X \mid x \land B \subseteq \{0\}\}$$

$$= \{x \in X \mid x \land b = 0, \ \forall b \in B\}$$

$$= B^* \quad (8)$$

which is the annihilator of $B$ (see Huang [15]). Hence the concept of the relative annihilator of $B$ with respect to $A$ is a generalization of the concept of the annihilator of $B$.

**Proposition 2.** [4] For any nonempty subsets $A$ and $B$ of $X$, we have

(i) If $A$ is an ideal of $X$, then $A \subseteq (A : \land B)$.

(ii) If $B_1 \subseteq B_2$ in $X$, then $(A : \land B_2) \subseteq (A : \land B_1)$.

**Lemma 1.** [4] If $A$ and $B$ are ideals of $X$, then the relative annihilator $(A : \land B)$ of $B$ with respect to $A$ is an ideal of $X$.

For any nonempty subsets $A$ and $B$ of $X$, let us now consider the set

$$F := \{x \in X \mid x \land B \subseteq A \land B\} \quad (9)$$

where $x \land B$ and $A \land B$ are meet sets. The following example shows that such a set need not be an ideal.
Example 3. Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the one in Example 2).

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We have five ideals of $X$: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$, and $A_4 = X$.

For subsets $A = \{2, 3\}$ and $B = \{4\}$ of $X$, we have

$$\{x \in X \mid x \land B \subseteq A \land B\} = \{1, 2, 3\},$$

which is not an ideal of $X$.

3 Some new results on ideals of lower BCK–semilattices

First of all, we provide conditions for the set $F = \{x \in X \mid x \land B \subseteq A \land B\}$ to be an ideal.

Theorem 2. If $A$ and $B$ are ideals of a lower BCK-semilattice $X$, then the set

$$F = \{x \in X \mid x \land B \subseteq A \land B\}$$

is an ideal of $X$.

Proof. Since $A$ and $B$ are ideals of $X$, using Theorem 1 shows that $A \land B$ is an ideal. So $0 \in A \land B$.

Also, $0 \land B = 0 \in A \land B$. Hence, $0 \in F$.

Let $x, y \in X$ be such that $x \ast y \in F$ and $y \in F$. Then $(x \ast y) \land B \subseteq A \land B$ and $y \land B \subseteq A \land B$, that is,

$$(x \ast y) \land b \in A \land B$$

and

$$y \land b \in A \land B$$

for all $b \in B$. Since $b \in B$, we have $\langle b \rangle \subseteq B$. It follows that $y \land \langle b \rangle \subseteq y \land B$ and $(x \ast y) \land \langle b \rangle \subseteq (x \ast y) \land B$. Thus, $y \land \langle b \rangle \subseteq A \land B$ and $(x \ast y) \land \langle b \rangle \subseteq A \land B$. It means that,

$$y \in (A \land B : \land \langle b \rangle)$$

and

$$x \ast y \in (A \land B : \land \langle b \rangle)$$

Using Lemma 1, we conclude that $(A \land B : \land \langle b \rangle)$ is an ideal of $X$. So (12), (13) and (3) show that

$$x \in (A \land B : \land \langle b \rangle)$$

Hence, or all $b \in B$, $x \land b \in x \land \langle b \rangle \subseteq A \land B$. Therefore,

$$x \land B \subseteq A \land B,$$

which means that $x \in F$. □
In the following text we are going to investigate properties of such ideals. The name, which we choose for them, stresses the fact that the set $A$ is reduced to a one element set.

**Definition 7.** For two ideals $A$ and $B$ of $X$, the ideal

$$ \{ x \in X \mid x \land B \subseteq A \land B \} $$

is called $A$-reduced meet ideal based on $B$ and denoted by $\frac{A \land B}{B}$.

The following example shows that the converse of Theorem 2 is not true in general, which means that there exist subsets $A$ and $B$ of $X$ which are not ideals, but $\frac{A \land B}{B}$ is an ideal of $X$.

**Example 4.** Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table (different from the ones in Example 2 or Example 3).

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We have four ideals of $X$: $A_0 = \{0\}$, $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2, 4\}$ and $A_3 = X$. For subsets $A = \{0, 1\}$ and $B = \{1, 2\}$ of $X$ which are not ideals, we have

$$ \frac{A \land B}{B} = \{0, 1, 3\}, $$

which is the ideal $A_1$ of $X$.

**Proposition 3.** For any ideals $A$ and $B$ of $X$, we have

(i) $A \subseteq \frac{A \land B}{B}$ and $B \subseteq \frac{A \land A \land B}{B}$.

(ii) $\frac{A \land X}{X} = A$ and $\frac{A \land A}{A} = X$.

(iii) $\frac{A \land B}{B} = X \iff B \subseteq A$.

**Proof.** (i) Suppose an arbitrary $x \in A$. Then $x \land B \subseteq A \land B$ for all $B \in \mathcal{I}(X)$. Thus $x \in \frac{A \land B}{B}$, and so $A \subseteq \frac{A \land B}{B}$.

Let $x \in B$ be an arbitrary element. For any arbitrary element $y \in \frac{A \land B}{B}$, we have $y \land B \subseteq A \land B$. Since $x \in B$, it follows that $y \land x \in A \land B$. Thus,

$$ x \land \frac{A \land B}{B} \subseteq A \land B. $$

Besides,

$$ A \land B \subseteq A = A \land \frac{A \land B}{B}. $$
Therefore, \( x \land \frac{A \land B}{B} \subseteq A \land \frac{A \land B}{B} \), implying that

\[
x \in A \land \frac{A \land B}{A \land B}.
\]

(ii) Let \( x \in A \). Then \( x \land X \subseteq A \land X \), and so \( x \in \frac{A \land X}{X} \). Conversely, let \( x \in \frac{A \land X}{X} \). Then \( x \land X \subseteq A \land X \). Since \( A \land X = A \) and \( x \land x = x \), we have \( x \in A \). Therefore,

\[
\frac{A \land X}{X} = A.
\]

Clearly,

\[
\frac{A \land A}{A} = \{x \in X \mid x \land A \subseteq A \land A\} = X.
\]

(iii) Suppose that \( \frac{A \land B}{B} = X \). Let \( b \) be an arbitrary element of \( B \). Since \( B \subseteq X \), clearly we have \( b \land B \subseteq A \land B \), and so

\[
b = b \land b \in A \land B.
\]

Thus \( B \subseteq A \land B \). Also, we always have \( A \land B \subseteq B \). Therefore, \( A \land B = B \) which means that \( B \subseteq A \).

Conversely, suppose that \( B \subseteq A \). Let \( x \in X \) and \( b \in B \) be an arbitrary element. Then \( x \land b \leq b \), and thus by using (4), \( x \land b \in B \subseteq A \), i.e. \( x \land b \in A \land B \) and \( x \in \frac{A \land B}{B} \). Thus \( X \subseteq \frac{A \land B}{B} \), and so \( X = \frac{A \land B}{B} \).

The following example shows that the converse of Proposition 3 (i), is not true in general, meaning that there exist ideals \( A \) and \( B \) of \( X \) such that \( \frac{A \land B}{B} \not\subseteq A \) and \( \frac{A \land B}{A \land B} \not\subseteq B \).

**Example 5.** Consider the lower BCK-semilattice \( X = \{0, 1, 2, 3, 4\} \) with the following Cayley table (again different from all the above examples).

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We have six ideals of \( X \): \( A_0 = \{0\} \), \( A_1 = \{0, 1\} \), \( A_2 = \{0, 1, 2\} \), \( A_3 = \{0, 1, 3\} \), \( A_4 = \{0, 1, 2, 3\} \) and \( A_5 = X \). For ideals \( A = A_2 = \{0, 1, 2\} \) and \( B = A_1 = \{0, 1\} \) of \( X \) we have

\[
\frac{A \land B}{B} = A_5,
\]

and clearly,

\[
\frac{A \land B}{B} \not\subseteq A.
\]
Also,
\[ A \land \frac{A \land B}{B} = A \land A_5 = A. \]

Thus,
\[ \{ x \in X \mid x \land \frac{A \land B}{B} \subseteq A \land \frac{A \land B}{B} \} = \{ x \in X \mid x \land A_5 \subseteq A \} = \{0, 1, 2\}, \]
and hence,
\[ A \land \frac{A \land B}{B} \not\subseteq B. \]

**Proposition 4.** For any ideals \( A \) and \( B \) of \( X \) such that \( A \subseteq B \), we have
\[ \frac{A \land B}{B} \cap B = A. \]

**Proof.** Let \( x \in A \). Then clearly \( x \in B \) and \( x \land B \subseteq A \land B \). Thus
\[ x \in \frac{A \land B}{B} \cap B. \]

Now suppose that \( x \in \frac{A \land B}{B} \cap B \). Then \( x \in B \) and \( x \in \frac{A \land B}{B} \). So \( x \land B \subseteq A \land B \). Since \( A \subseteq B \) and \( x \in B \), we have \( A \land B = A \) and
\[ x = x \land x \in A \land B = A. \]

Therefore, \( x \in A \) and
\[ \frac{A \land B}{B} \cap B = A. \]

\[ \square \]

**Proposition 5.** Let \( B \) be an ideal of \( X \). For any family \( \{ A_\lambda \mid \lambda \in \Lambda \} \) of ideals, we have
\[ \left( \bigcap_{\lambda \in \Lambda} \frac{A_\lambda \land B}{B} \right) = \bigcap_{\lambda \in \Lambda} \left( \frac{A_\lambda \land B}{B} \right). \] (14)

**Proof.** Let \( x \in \left( \bigcap_{\lambda \in \Lambda} \frac{A_\lambda \land B}{B} \right) \). Then \( x \land B \subseteq \bigcap_{\lambda \in \Lambda} A_\lambda \land B \). Hence there exist \( a \in \bigcap_{\lambda \in \Lambda} A_\lambda \) and \( b \in B \) such that \( x \land y = a \land b \). Thus, for every \( \lambda \in \Lambda \), \( a \land b \in A_\lambda \land B \) and so \( x \land y \in A_\lambda \land B \). Since \( y \) is an arbitrary element of \( B \), we have \( x \land B \subseteq A_\lambda \land B \). Hence, for every \( \lambda \in \Lambda \),
\[ x \in \left( \frac{A_\lambda \land B}{B} \right) \]
which means that
\[ x \in \bigcap_{\lambda \in \Lambda} \left( \frac{A_\lambda \land B}{B} \right). \]

Therefore,
\[ \left( \bigcap_{\lambda \in \Lambda} \frac{A_\lambda \land B}{B} \right) \subseteq \bigcap_{\lambda \in \Lambda} \left( \frac{A_\lambda \land B}{B} \right). \]
Now suppose that \( x \in \cap_{\lambda \in \Lambda} \left( A_{\lambda} \wedge B \right) \). Then for every \( \lambda \in \Lambda \), \( x \in A_{\lambda} \wedge B \) and so \( x \cap B \subseteq A_{\lambda} \wedge B \). For an arbitrary element \( y \in B \) and every \( \lambda \in \Lambda \), \( x \cap y \in A_{\lambda} \wedge B \). So for \( \lambda_1 \in \Lambda \), there exist \( a_{\lambda_1} \in A_{\lambda_1} \) and \( b_i \in B \) such that \( x \cap y = a_{\lambda_1} \wedge b_i \). Similarly, for \( \lambda_i \in \Lambda \), \( 2 \leq i \leq n \), there exist \( a_{\lambda_i} \in A_{\lambda_i} \) and \( b_i \in B \) such that \( x \cap y = a_{\lambda_i} \wedge b_i \). Now we have

\[
x \cap y = (x \cap y) \wedge (x \cap y) \wedge \ldots \wedge (x \cap y)
\]

\[
= (a_{\lambda_1} \wedge b_i) \wedge (a_{\lambda_2} \wedge b_i) \wedge \ldots \wedge (a_{\lambda_n} \wedge b_i)
\]

\[
= (a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n}) \wedge (b_1 \wedge b_2 \ldots \wedge b_n)
\]

Clearly, \( a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n} \in \bigcap_{\lambda \in \Lambda} A_{\lambda} \) and \( b_1 \wedge b_2 \ldots \wedge b_n \in B \). Using Proposition 1, we conclude that

\[
a_{\lambda_1} \wedge a_{\lambda_2} \wedge \ldots \wedge a_{\lambda_n} \in \bigcap_{\lambda \in \Lambda} A_{\lambda}.
\]

Thus, \( x \cap y \in \bigcap_{\lambda \in \Lambda} A_{\lambda} \cap B \). Since \( y \) is an arbitrary element of \( X \), we have

\[
x \in \left( \bigcap_{\lambda \in \Lambda} A_{\lambda} \cap B \right).
\]

Therefore,

\[
\left( \bigcap_{\lambda \in \Lambda} A_{\lambda} \cap B \right) = \bigcap_{\lambda \in \Lambda} \left( A_{\lambda} \cap B \right).
\]

\[\square\]

**Theorem 3.** For any ideals \( A \) and \( B \) of \( X \), we have

\( A \cap B \subseteq \frac{A \cap B}{B} \subseteq \frac{A :_\Lambda B}{B} \). \hspace{1cm} (15)

Moreover, if \( A \subseteq B \), then

\( \frac{A \cap B}{B} = (A :_\Lambda B) \). \hspace{1cm} (16)

**Proof.** Let \( x \in A \cap B \) be an arbitrary element. Then there exist \( a \in A \) and \( b \in B \) such that \( x = a \wedge b \). For an arbitrary element \( b' \in B \) we have

\[
x \cap b' = (a \wedge b) \wedge b' = a \wedge (b \wedge b').
\]

Since \( b \wedge b' \leq b \) and \( b \in B \), using (4) we conclude that \( b \wedge b' \in B \). Thus, \( a \wedge (b \wedge b') \in A \wedge B \). Therefore, \( A \wedge B \subseteq \frac{A \wedge B}{B} \).

Now let \( x \in \frac{A \wedge B}{B} \). Then for every element \( b \in B \), \( x \cap b \in A \wedge B \). Since \( A \wedge B \subseteq A \), we have \( x \cap b \in A \). Hence \( x \wedge B \subseteq A \) and \( x \in \left( A :_\Lambda B \right) \). Therefore, \( \frac{A \wedge B}{B} \subseteq \frac{A :_\Lambda B}{B} \).

If \( A \subseteq B \), then \( A \wedge B = A \) and equation (16) is clear. \[\square\]

**Theorem 4.** For ideals \( A, B_1 \) and \( B_2 \) of \( X \), if \( B_1 \subseteq B_2 \), then

\[
\frac{A \wedge (A :_\Lambda B_1)}{(A :_\Lambda B_1)} \subseteq \frac{A \wedge (A :_\Lambda B_2)}{(A :_\Lambda B_2)}.
\]

(17)
Proof. Suppose that \( B_1 \subseteq B_2 \) and \( x \in \frac{A \land (A : \land B_1)}{(A : \land B_2)} \). Then

\[
x \land (A : \land B_1) \subseteq A \land (A : \land B_1).
\]

Using Proposition 2, we have

\[
A \land (A : \land B_1) = A = A \land (A : \land B_2),
\]

\[
(A : \land B_2) \subseteq (A : \land B_1).
\]

Thereby,

\[
x \land (A : \land B_2) \subseteq x \land (A : \land B_1) \subseteq A \land (A : \land B_1) = A = A \land (A : \land B_2).
\]

Therefore, \( x \land (A : \land B_2) \subseteq A \land (A : \land B_2) \) and thus

\[
x \in \frac{A \land (A : \land B_2)}{(A : \land B_2)}.
\]

The following example shows that the converse of inclusion (17) is not true in general, which means that there exist ideals \( A, B_1 \) and \( B_2 \) of \( X \) such that \( B_1 \subseteq B_2 \), but

\[
\frac{A \land (A : \land B_2)}{(A : \land B_2)} \nsubseteq \frac{A \land (A : \land B_1)}{(A : \land B_1)}
\]

**Example 6.** Consider the lower \( BCK \)-semilattice \( X = \{0, 1, 2, 3, 4\} \) with the following Cayley table.

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

For ideals \( A = \{0, 1\} \), \( B_1 = \{0, 3\} \) and \( B_2 = \{0, 2, 3\} \) of \( X \), we have

\[
(A : \land B_1) = \{x \in X \mid x \land B_1 \subseteq A\} = \{0, 1, 2, 4\},
\]

and

\[
(A : \land B_2) = \{x \in X \mid x \land B_2 \subseteq A\} = \{0, 1, 4\}.
\]

Thus,

\[
\frac{A \land (A : \land B_1)}{(A : \land B_1)} = \{x \in X \mid x \land (A : \land B_1) \subseteq A \land (A : \land B_1)\} = \{0, 1\},
\]

and

\[
\frac{A \land (A : \land B_2)}{(A : \land B_2)} = \{x \in X \mid x \land (A : \land B_2) \subseteq A \land (A : \land B_2)\} = \{0, 1, 2, 3\}.
\]
Clearly,
\[
\frac{A \land (A : \land B_1)}{(A : \land B_1)} \subseteq \frac{A \land (A : \land B_2)}{(A : \land B_2)}.
\]

But
\[
\frac{A \land (A : \land B_2)}{(A : \land B_2)} \not\subseteq \frac{A \land (A : \land B_1)}{(A : \land B_1)}.
\]

**Proposition 6.** For ideals \( A, B_1 \) and \( B_2 \) of \( X \), we have
\[
\frac{A \land (A : \land B_1)}{(A : \land B_1)} \cap \frac{A \land (A : \land B_2)}{(A : \land B_2)} \subseteq \frac{A \land (A : \land \langle B_1 \cup B_2 \rangle)}{(A : \land \langle B_1 \cup B_2 \rangle)}.
\]

**Proof.** Since \( B_1 \subseteq \langle B_1 \cup B_2 \rangle \), using Theorem 4, we conclude that
\[
\frac{A \land (A : \land B_1)}{(A : \land B_1)} \subseteq \frac{A \land (A : \land \langle B_1 \cup B_2 \rangle)}{(A : \land \langle B_1 \cup B_2 \rangle)}.
\]

Similarly, we have
\[
\frac{A \land (A : \land B_2)}{(A : \land B_2)} \subseteq \frac{A \land (A : \land \langle B_1 \cup B_2 \rangle)}{(A : \land \langle B_1 \cup B_2 \rangle)}.
\]

Therefore,
\[
\frac{A \land (A : \land B_1)}{(A : \land B_1)} \cap \frac{A \land (A : \land B_2)}{(A : \land B_2)} \subseteq \frac{A \land (A : \land \langle B_1 \cup B_2 \rangle)}{(A : \land \langle B_1 \cup B_2 \rangle)}.
\]

\[\square\]

### 4 Future work

The roots of \( BCI- \), \( BCK- \), \( MV- \) and other types of algebras lie in combinatory logic and information sciences. Notice that recently Jun and Song [18] and Flaut [13] linked \( BCK- \)–algebras to block codes used in channel encoding in earlier mobile communication systems. Matrices of block codes are studied in Saeid et al. [21]. Since, in a special case, we obtain lattices (or semilattices), whenever multivalued aspects are employed we can make use of results of the hyperstructure theory and concepts such as e.g. \( EL- \)–hyperstructures (see e.g. Křehlík and Novák [19] where sets of matrices and lattices are studied) or even classical results of Varlet [22] who provided the link between distributive lattices and hyperstructure theory. In our future work we shall concentrate on finding links between our results and the above mentioned concepts, or those connected with closure operations on \( BCK- \)–algebras [1, 2].

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