Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

CONTROL SMOOTHING SPLINES WITH INITIAL CONDITIONS

ASMUSS Svetlana (LV), BUDKINA Natalja (LV)

Abstract. This work is devoted to the problem of optimal control of a linear dynamic system with initial conditions. The main attention is paid to a controlled system reduced to the second order differential equation considered with the cost functional which controls the input function of this system to push a trajectory nearby a set of desired points. We show how the technique of smoothing splines can be adapted for construction of solutions of such problem.

Keywords: smoothing spline, control theory, dynamic system

Mathematics subject classification: Primary 65D07; Secondary 49J15, 65K10

1 Introduction

This work is devoted to the problem of determining control and state trajectories for a linear dynamic system over a period of time to minimize an objective function. To solve this problem in some special cases we use the results from the theory of splines. The obtained solutions are so called control theoretic smoothing splines whose characteristics depend on the dynamics of the control system.

Spline functions are well known and widely used for practical approximation of functions by the information of values of function at the given points. The smoothing spline is a smooth function s in a suitable function space that minimizes the objective functional with a weight parameter which controls the smoothing. We refer the reader to [4] and references therein for the properties of smoothing splines. From the control theory point of view, the smoothing spline model is closely related to the finite-horizon linear quadratic optimal control problem by treating derivatives of s as a control input. There are a lot of publications about relations between control theory and smoothing splines, which in this context called smoothing theoretic splines. It should be mentioned also, that in the most of the papers (see, e.g., [2],[3],[5]) about control splines the problems on splines are reduced to the problem of control theory. The aim of this paper is otherwise to reduce the problem of control theory to the problem of smoothing splines and to use for its analysis methods and results of the general theory of splines. It will be done for some special cases of the following control theory problem:

$$x' = Mx + \beta u, \;\; y = \gamma^\top x, \tag{1}$$

considered with additional initial condition $x(a) = \alpha$.

We consider system (1) as the curve $z = y(t)$ generator. Our aim is to find a control law $u \in L_2[a, b]$ which drives the scalar output trajectory close to a finite sequence of set points at fixed times

$$\{(t_i, z_i) : i = 1, 2, \dots, n\}, \text{ where } a < t_1 < t_2 < \dots < t_n < b.$$

and minimizes the objective functional with a positive weight ρ:

$$\int_a^b u^2(t)dt + \rho \sum_{i=1}^{n} (y(t_i) - z_i)^2 \to min.$$

The main attention in this paper is paid to the special case of problem (1):

$$M = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

For this case the objective functional of the problem of optimal control could be rewritten as

$$\int_a^b (x_1''(t) + px_1'(t) + qx_1(t))^2 dt + \rho \sum_{i=1}^{n} (\gamma_1 x_1(t_i) + \gamma_2 x_1'(t_i) - z_i)^2 \longrightarrow \min_{x_1(a)=\alpha_1, x_1'(a)=\alpha_2}. \quad (2)$$

We also note that usually such problem has been considered without initial conditions.

2 Smoothing splines

Problem (2) corresponds to the following more general conditional minimization problem:

$$||Tg||^2 + \rho ||Ag - z||^2 \longrightarrow \min_{Bg=\alpha}, \quad (3)$$

where linear operators $T : W_2^r[a, b] \to L_2[a, b]$, $A : W_2^r[a, b] \to I\!R^n$ and $B : W_2^r[a, b] \to I\!R^m$ are continuous, parameter $\rho > 0$ and vectors $z \in I\!R^n$ and $\alpha \in I\!R^m$ are given. We assume that $A(W_2^r[a, b]) = I\!R^n$, $B(W_2^r[a, b]) = I\!R^m$ and all functionals of A and B, i.e. A_i, B_j, $i = 1, \dots, n$, $j = 1, \dots, m$, are linearly independent and we recall the theorem (see, e.g., [1]) on the existence and characterization of solutions of problem (3).

Theorem 1 *Under the assumptions that $kerT + kerA$ is closed a solution of problem* (3) *exists. An element $s \in B^{-1}(\alpha)$ is a solution of this problem if and only if there exist vectors $\lambda \in I\!R^n$ and $\nu \in I\!R^m$ such that*

$$T^*Ts = A^*\lambda + B^*\nu \text{ and } \lambda + \rho(As - z) = 0. \quad (4)$$

Under the additional assumption $kerT \cap kerB = \{0\}$ this theorem gives the uniqueness of solution.

The characterization theorem implies that a solution of problem (3) is a spline from the space

$$S(T, (A, B)) = \{s \in W_2^r[a, b] : \ < Ts, Tg >= 0 \text{ for all } g \in ker(A, B)\},$$

where (A, B) consists of all functionals of A and B. Here and in the sequel $<, >$ is the scalar product and $kerA$ is the kernel of operator A.

Let us denote by $P : W_2^r[a,b] \to kerT$ a projector to the kernel of T. By using P we rewrite the first equation of (4) as

$$< Ts, Tg >= \sum_{i=1}^{n} \lambda_i A_i(g - Pg) + \sum_{j=1}^{m} \nu_j B_j(g - Pg). \tag{5}$$

For applying (5) in this paper we consider $g - Pg$ written in an integral form which includes Tg:

$$g(t) = (Pg)(t) + \int_a^b (Tg)(\tau)K(t,\tau)d\tau. \tag{6}$$

Now we rewrite (5) in the form

$$\int_a^b (Tg)(\tau)(Ts)(\tau)d\tau = \int_a^b (Tg)(\tau)(\sum_{i=1}^{n} \lambda_i A_i(K(.,\tau)) + \sum_{j=1}^{m} \nu_j B_j(K(.,\tau)))d\tau$$

(here $A(K(.,\tau))$ means that $K(\cdot,\tau)$ is considered as the function of the first argument when the second argument is fixed as τ) which implies

$$(Ts)(\tau) = \sum_{i=1}^{n} \lambda_i A_i(K(.,\tau)) + \sum_{j=1}^{m} \nu_j B_j(K(.,\tau)) \text{ and}$$

$$s(t) = (Ps)(t) + \sum_{i=1}^{n} \lambda_i \int_a^b A_i(K(.,\tau))K(t,\tau)d\tau + \sum_{j=1}^{m} \nu_j \int_a^b B_j(K(.,\tau))K(t,\tau)d\tau.$$

Let us assume that $dim(kerT) = m$, $kerT \cap kerB = \{0\}$ and $g - Pg \in kerB$ for all g. Then the unique solution of problem (3) is

$$s(t) = (Ps)(t) + \sum_{i=1}^{n} \lambda_i \int_a^b A_i(K(.,\tau))K(t,\tau)d\tau \tag{7}$$

such that

$$A_i s + \frac{\lambda_i}{\rho} = z_i, \; i = 1, \ldots, n, \; \text{ and } \; B_j Ps = \alpha_j, \; j = 1, \ldots, m. \tag{8}$$

Let us note that vector $\nu \in I\!R^m$ (see Theorem 1) is such that

$$\sum_{i=1}^{n} \lambda_i A_i h + \sum_{j=1}^{m} \nu_j B_j h = 0 \;\text{ for all }\; h \in kerT. \tag{9}$$

3 Smoothing splines in control theory

We consider problem (2) as problem (3) with

$$Tg = g'' + pg' + qx_1 = u, \quad A_i g = \gamma_1 g(t_i) + \gamma_2 g'(t_i), i = 1, ..., n, \quad B_1 g = g(a), B_2 g = g'(a).$$

In this case operators T, A and B fulfill all assumptions mentioned in the previous section. In the sequel we obtain the view of solution of problem (2) and on this basis we describe special classes of control splines from $S(T, (A, B))$ depending on the roots r_1, r_2 of the equation $r^2 + pr + q = 0$.

- Class 1 (exponential splines with polynomial coefficients): $r_1 = r_2 \in I\!R \setminus \{0\}$.
- Class 2 (exponential splines): $r_1, r_2 \in I\!R, \ r_1 \neq r_2$.
- Class 3 (polynomial-exponential splines): $r_1, r_2 \in I\!R, \ r_1 \neq r_2, r_1 \neq 0, r_2 = 0$.
- Class 4 (polynomial splines): $r_1 = r_2 = 0$.
- Class 5 (trigonometric splines with polynomial coefficients): $r_{1,2} = \pm i\eta \neq 0$.
- Class 6 (trigonometric splines with exponential-polynomial coefficients): $r_{1,2} = \zeta \pm i\eta$.

The remaining part of this section is organized as follows: the subsections correspond to the classes mentioned above; we present a proof of the obtained result for the first class (Subsection 3.1) and give results for other classes (Subsections 3.2-3.6) without proofs due to the space limitation.

3.1 Exponential splines with polynomial coefficients

We consider the case $r_1 = r_2 \in I\!R \setminus \{0\}$, i.e., $q > 0, \ p^2 = 4q$ and $Tg = g'' - 2r_1 g' + r_1^2 g$. The kernel of operator T is $kerT = \{(C_1 + C_2 t)e^{r_1 t} \mid C_1, C_2 \in I\!R\}$. We use (6) with

$$(Pg)(t) = g(a)e^{r_1(t-a)} + (g'(a) - r_1 g(a))e^{r_1(t-a)}(t-a), \quad K(t, \tau) = e^{r_1(t-\tau)}(t-\tau)_+$$

and obtain the following representation

$$s(t) = (\mu_1 + \mu_2(t-a))e^{r_1(t-a)} + \sum_{i=1}^{n} \lambda_i \int_a^b e^{r_1(t_i+t-2\tau)}((\gamma_1 + \gamma_2 r_1)(t_i - \tau)_+ + \gamma_2(t_i - \tau)_+^0)(t-\tau)_+ d\tau$$

for solution s of problem (2) by using (7). Applying (8) we obtain

$$\mu_1 = B_1 Ps = \alpha_1, \quad \mu_2 = B_2 Ps - r_1 B_1 Ps = \alpha_2 - r_1 \alpha_1, \tag{10}$$

$$s(t_i) + \frac{\lambda_i}{\rho} = z_i, i = 1, \dots, n. \tag{11}$$

The expression of Ts gives the corresponding control function u :

$$u(t) = (Ts)(t) = \sum_{i=1}^{n} \lambda_i e^{r_1(t_i - t)}((\gamma_1 + \gamma_2 r_1)(t_i - t)_+ + \gamma_2(t_i - t)_+^0). \tag{12}$$

By using the equalities $(t_i-\tau)_+^k=(t_i-\tau)^k+(-1)^{k+1}(\tau-t_i)_+$ and

$$(\gamma_1+\gamma_2 r_1)\sum_{i=1}^{n}\lambda_i e^{r_1 t_i}+(\nu_1+\nu_2 r_1)e^{r_1 a}=0,$$
$$(\nu_1 a+\nu_2 r_1 a+\nu_2)e^{r_1 a}+\sum_{i=1}^{n}\lambda_i((\gamma_1+\gamma_2 r_1)t_i e^{r_1 t_i}+\gamma_2 e^{r_1 t_i})=0, \tag{13}$$

(two last equalities are written using (9) with $h(t)=e^{r_1 t}$ and $h(t)=te^{r_1 t}$) we rewrite s:

$$s(t)=(\mu_1+\mu_2(t-a))e^{r_1(t-a)}+\int_a^t e^{r_1(a+t-2\tau)}((\nu_1+\nu_2 r_1)(\tau-a)-\nu_2))(t-\tau)d\tau+$$
$$+\sum_{i=1}^{n}\lambda_i\int_a^t e^{r_1(t_i+t-2\tau)}((\gamma_1+\gamma_2 r_1)(\tau-t_i)_+-\gamma_2(\tau-t_i)_+^0)(t-\tau)d\tau.$$

The integration gives us the result:

$$s(t)=(\mu_1+\mu_2(t-a))e^{r_1(t-a)}+\frac{\nu_1}{4r_1^3}(r_1(t-a)(e^{r_1(a-t)}+e^{r_1(t-a)})+(e^{r_1(a-t)}-e^{r_1(t-a)}))+$$
$$+\frac{\nu_2(t-a)}{4r_1}(e^{r_1(a-t)}-e^{r_1(t-a)})+\sum_{i=1}^{n}\lambda_i(\frac{\gamma_1}{4r_1^3}(r_1(t-t_i)_+(e^{r_1(t_i-t)}+e^{r_1(t-t_i)})+$$
$$+(e^{r_1(t_i-t)}-e^{r_1(t-t_i)})(t-t_i)_+^0)+\frac{\gamma_2}{4r_1}(e^{r_1(t_i-t)}-e^{r_1(t-t_i)})(t-t_i)_+). \tag{14}$$

The following proposition is proved.

Proposition 1 *In the case* $r_1=r_2\in I\!R\setminus\{0\}$ *the unique solution of problem* (2) *is exponential spline* (14) *which polynomial coefficients fulfill conditions* (10), (11), (13). *The corresponding control function* u *is given by* (12).

3.2 Exponential splines

In the case $r_1,r_2\in I\!R,\ r_1\neq r_2, r_1\neq 0, r_2\neq 0$ we use $Tg=g''-(r_1+r_2)g'+r_1r_2g$. The kernel of this operator is $kerT=\{C_1e^{r_1t}+C_2e^{r_2t}\mid C_1,C_2\in I\!R\}$.

By analogy with the previous case we obtain the solution s of problem (2) in the following form:

$$s(t)=\frac{\alpha_2-\alpha_1 r_2}{r_1-r_2}e^{r_1(t-a)}-\frac{\alpha_2-\alpha_1 r_1}{r_1-r_2}e^{r_2(t-a)}+\frac{1}{2(r_1^2-r_2^2)}(\frac{(\nu_1-\nu_2 r_1)e^{r_1(t-a)}}{r_1}-\frac{(\nu_1-\nu_2 r_2)e^{r_2(t-a)}}{r_2}+$$
$$+\sum_{i=1}^{n}\lambda_i(\frac{\gamma_1}{r_1}e^{r_1|t-t_i|}+\gamma_2(e^{r_1(t_i-t)_+}-e^{r_1(t-t_i)_+})-\frac{\gamma_1}{r_2}e^{r_2|t-t_i|}-\gamma_2(e^{r_2(t_i-t)_+}-e^{r_2(t-t_i)_+}))), \tag{15}$$

where coefficients ν_1 and ν_2 are expressed using the following system:

$$(\gamma_1+\gamma_2 r_1)\sum_{i=1}^{n}\lambda_i e^{r_1 t_i}+(\nu_1+\nu_2 r_1)e^{r_1 a}=0,\quad (\gamma_1+\gamma_2 r_2)\sum_{i=1}^{n}\lambda_i e^{r_2 t_i}+(\nu_1+\nu_2 r_2)e^{r_2 a}=0. \tag{16}$$

Proposition 2 *In the case $r_1, r_2 \in I\!R,\ r_1 \neq r_2, r_1 \neq 0, r_2 \neq 0$ the unique solution of problem* (2) *is exponential spline* (15) *which coefficients fulfill conditions* (11), (16). *The corresponding control function u is given by*

$$u(t) = \sum_{i=1}^{n} \frac{\lambda_i (t_i - t)_+^0}{r_1 - r_2} (\gamma_1 (^{r_1(t_i - t)} - e^{r_2(t_i - t)}) + \gamma_2 (r_1 e^{r_1(t_i - t)} - r_2 e^{r_2(t_i - t)})). \tag{17}$$

3.3 Polynomial-exponential splines

Now we consider the case when one of the roots $r_1, r_2 \in I\!R$ is equal to zero. The following result can be obtained by minimal changes of the previous proof.

Proposition 3 *In the case $r_1, r_2 \in I\!R,\ r_1 \neq r_2, r_1 \neq 0, r_2 = 0$ the unique solution of problem* (2) *is polynomial-exponential spline*

$$s(t) = \tfrac{\alpha_2}{r_1} e^{r_1(t-a)} + \alpha_1 - \tfrac{\alpha_2}{r_1} + \tfrac{1}{r_1^3}((\nu_1 - \nu_2 r_1) e^{r_1(t-a)} - (\nu_1 + \nu_2 r_1) e^{r_1(a-t)} + 2 r_1 \nu_2 - 2 r_1 \nu_1 (t-a) +$$

$$+ \sum_{i=1}^{n} \lambda_i (t - t_i)_+^0 (-\gamma_1 r_1 (t - t_i) - \tfrac{\gamma_1 + \gamma_2 r_1}{2} e^{r_1(t_i - t)} + \tfrac{\gamma_1 - \gamma_2 r_1}{2} e^{r_1(t - t_i)} + \gamma_2 r_1))$$

which coefficients fulfill (11) *and the following system*

$$(\gamma_1 + \gamma_2 r_1) \sum_{i=1}^{n} \lambda_i e^{r_1 t_i} + (\nu_1 + \nu_2 r_1) e^{r_1 a} = 0, \quad \sum_{i=1}^{n} \lambda_i \gamma_1 + \nu_1 = 0,$$

The corresponding control function u is given by (17).

3.4 Cubic splines

In the case $r_1 = r_2 = 0$, i.e., $p = q = 0$, we use $Tg = g''$. This case corresponds to the classical smoothing problem in the theory of splines according to which a solution of (2) without additional conditions is a cubic spline. Taking into account the initial conditions and Theorem 1 we get the solution s of problem (2) and the corresponding control function u:

$$s(t) = \alpha_1 + \alpha_2 (t-a) + \tfrac{\nu_1}{6} (t-a)_+^3 - \tfrac{\nu_2}{2} (t-a)_+^2 + \sum_{i=1}^{n} \tfrac{\lambda_i}{6} (t - t_i)_+^3, \quad u(t) = s''(t),$$

with the coefficients which fulfill the conditions (11) and $\nu_1 + \sum_{i=1}^{n} \lambda_i = 0, \quad \nu_1 t_0 + \nu_2 + \sum_{i=1}^{n} \lambda_i t_i = 0.$

3.5 Trigonometric splines with polynomial coefficients

Now we consider the case when $r_{1,2} = \pm i\eta \neq 0$, i.e., $p = 0,\ q > 0,\ \eta = \sqrt{q}$. It means that we take $Tg = g'' + qg$. The kernel of this operator is $kerT = \{C_1 cos\, \eta t + C_2 sin\, \eta t \mid C_1, C_2 \in I\!R\}$.

Proposition 4 *In the case $r_{1,2} = \pm\eta \neq 0$ the unique solution of problem* (2) *is trigonometric spline*

$$s(t) = \alpha_1 cos\,(\eta(t-a)) + \frac{\alpha_2}{\eta} sin\,(\eta(t-a)) + \frac{\nu_1}{2\eta^3} (\eta(t-a) cos\,(\eta(t-a)) - sin\,(\eta(t-a))) +$$

$+\frac{\nu_2}{2\eta}(t-a)sin\,(\eta(t-a))+\frac{1}{2}\sum_{i=1}^{n}\lambda_i(\frac{\gamma_1}{\eta^3}sin\,(\eta(t-t_i)_+)-\frac{(t-t_i)_+}{\eta^2}(\gamma_1 cos\,(\eta(t-t_i))+\gamma_2\eta sin\,(\eta(t-t_i)))$

which polynomial coefficients fulfill (11) and

$$\sum_{i=1}^{n}\lambda_i(\gamma_1 sin\,(\eta t_i)+\gamma_2\eta cos\,(\eta t_i))+\nu_1 sin\,(\eta a)+\nu_2\eta cos\,(\eta a)=0,$$
$$\sum_{i=1}^{n}\lambda_i(\gamma_1 cos\,(\eta t_i)-\gamma_2\eta sin\,(\eta t_i))+\nu_1 cos\,(\eta a)-\nu_2\eta sin\,(\eta a)=0.$$

The corresponding control function u is given by

$$u(t)=\sum_{i=1}^{n}\lambda_i(\frac{\gamma_1}{\eta}sin\,\eta(t_i-t)_+ +\gamma_2(t_i-t)_+^0 cos\,\eta(t_i-t)).$$

3.6 Trigonometric splines with exponential-polynomial coefficients

In this section we consider the case $r_{1,2}=\zeta\pm i\eta$, i.e., $p^2-4q<0, \zeta\neq 0, \eta\neq 0$. We use operator $Tg=g''-2\zeta g'+(\zeta^2+\eta^2)g$ with the kernel $kerT=\{C_1e^{\zeta t}cos\,\eta t+C_2e^{\zeta t}sin\,\eta t\mid C_1,C_2\in IR\}$.

Proposition 5 *In the case $r_{1,2}=\zeta\pm i\eta, \zeta\neq 0, \eta\neq 0$ the unique solution of problem* (2) *is trigonometric spline*

$$s(t)=\alpha_1 e^{\zeta(t-a)}cos\,(\eta(t-a))+\frac{\alpha_2-\zeta\alpha_1}{\eta}e^{\zeta(t-a)}sin\,(\eta(t-a))+\sum_{i=1}^{n}\frac{\lambda_i(\tau-t_i)_+^0}{4\zeta\eta}((e^{\zeta(t_i-t)}-e^{\zeta(t-t_i)})\times$$
$$\times(\frac{\gamma_1\eta}{\zeta^2+\eta^2}cos\,\eta(t-t_i)+\gamma_2 sin\,\eta(t-t_i))+(e^{\zeta(t_i-t)}+e^{\zeta(t-t_i)})sin\,\eta(t-t_i)\frac{\gamma_1\zeta}{\zeta^2+\eta^2}+\frac{1}{4\eta}((e^{\zeta(a-t)}-$$
$$-e^{\zeta(t-a)})(\frac{\nu_1\eta}{\zeta^2+\eta^2}cos\,\eta(t-a)+\nu_2 sin\,\eta(t-a))+(e^{\zeta(a-t)}+e^{\zeta(t-a)})sin\,\eta(t-a)\frac{\zeta\nu_1}{\zeta^2+\eta^2}).$$

which exponential-polynomial coefficients fulfill (11) *and*

$$\sum_{i=1}^{n}\lambda_i e^{\zeta t_i}((\gamma_1+\gamma_2\zeta)sin\,\eta t_i+\gamma_2\eta cos\,\eta t_i)+e^{\zeta a}((\nu_1+\nu_2\zeta)sin\,\eta a+\nu_2\eta cos\,\eta a)=0,$$
$$\sum_{i=1}^{n}\lambda_i e^{\zeta t_i}((\gamma_1+\gamma_2\zeta)cos\,\eta t_i-\gamma_2\eta sin\,\eta t_i)+e^{\zeta a}((\nu_1+\nu_2\zeta)cos\,\eta a-\nu_2\eta sin\,\eta a)=0.$$

The corresponding control function u is given by

$$u(t)=\sum_{i=1}^{n}\frac{\lambda_i(t_i-t)_+^0}{\eta}e^{\zeta(t_i-t)}((\gamma_1+\gamma_2\zeta)sin\,\eta(t_i-t)+\gamma_2\eta cos\,\eta(t_i-t)).$$

4 Numerical example

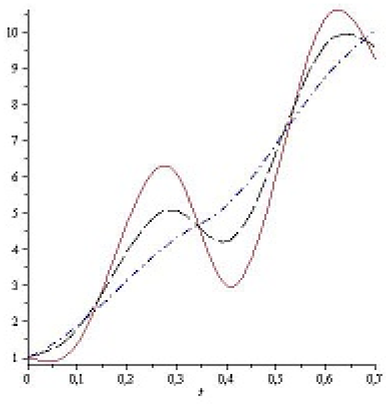
We consider the problem (2) with $\gamma_1=1$, $\gamma_2=0$, $\alpha_1=1$, $\alpha_2=1$, $p=-3$, $q=2$, i.e., $r_1=1, r_2=2$, and desired points: $(0.1;1), (0.3;7), (0.4;2), (0.6;11), (0.7;9)$. We solve this problem on $[0:0.7]$ with three different values of ρ: $\rho_1=1000$, $\rho_2=7000$ and $\rho_3=30000$.

In this case the unique solution of (2) is the exponential spline (15) with coefficients from Tab.1. We note that $s(t_i)-z_i=\lambda_i/\rho, i=1,...,n$.

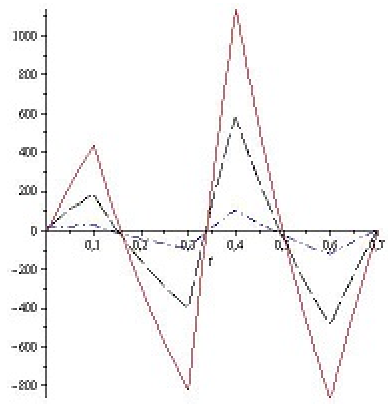
The graphs of s for ρ_1 (dash dot line), ρ_2 (dash long line) and ρ_2 (solid line) are shown in Figure 1 (a), the graphs of the corresponding control function u are given in Figure 1 (b).

	λ_1/ρ	λ_2/ρ	λ_3/ρ	λ_4/ρ	λ_5/ρ	ν_1	ν_2
ρ_1	-0.859	2.696	-3.203	2.244	-1.063	155.34	-16.64
ρ_2	-0.772	1.926	-2.215	1.359	-0.602	2064.43	-7.35
ρ_3	-0.408	0.911	-1.006	0.578	-0.251	5181.63	12.86

Tab. 1. Coefficients of the solution.



(a)



(b)

Fig. 1. Graphs of solution.

References

[1] ASMUSS, S., BUDKINA, N.: On Some Generalization of Smoothing Problems. *Mathematical Modelling and Analysis*, V. 15, Nr. 3, 2015: 11-28.

[2] EGERSTEDT, M., MARTIN, C.: *Control Theoretic Splines*, Princeton University Press, 2010.

[3] SHEN, J., LEBAIR, J., MARTIN, C.: Splines and Linear Control Theory. *Automatica*, V. 53, 2015: 216-224.

[4] WAHBA, G.: *Spline Models for Observational Data*, in CBMS-NSF Regional Conference Series in Applied Mathematics, Philadelphia: SIAM, 1990.

[5] ZHANG, Z., TOMLINSON, T. M.: Shape Restricted Smoothing Splines via Constrained Optimal Control and Nonsmooth Newton's Methods. In *Acta Applicandae Mathematica*, V. 49, Nr. 1, 1997: 1–34.

Current address

Asmuss Svetlana, Dr. Math.
University of Latvia
Zellu street 25, Riga, LV-1002, Latvia
Institute of Mathematics and Computer Science of University of Latvia
Rainis blvd. 29, Riga, LV-1459, Latvia
E-mail: svetlana.asmuss@lu.lv

Budkina Natalja, Dr. Math.
Faculty of Computer Science and Information Technology
Riga Technical University
Daugavgrivas street 2, Riga, LV-1048, Latvia
E-mail: natalja.budkina@rtu,lv